Inhomogeneous Phases in the NJL Model with Vector Interactions

NeD/TURIC Workshop 2014

Marco Schramm

The QCD Phase Diagram

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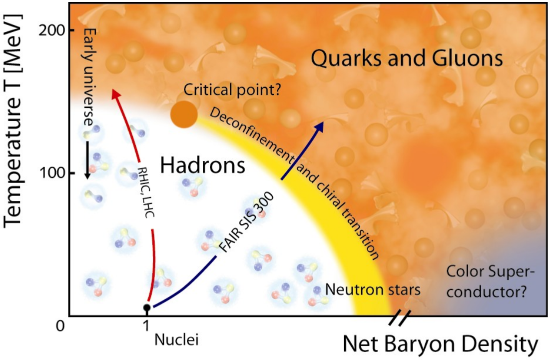

(GSI)

Effective Model

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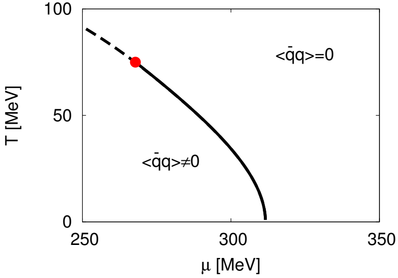
Nambu–Jona-Lasinio model

- Shares chiral symmetry with QCD
- Order parameter: chiral condensate $\langle \overline{q} q \rangle$
- Related to constituent quark mass
- In this work: include vector interactions
    - important in similar models for nuclear matter (Walecka model)
    - needed for description of vector mesons



Inhomogeneous Phase

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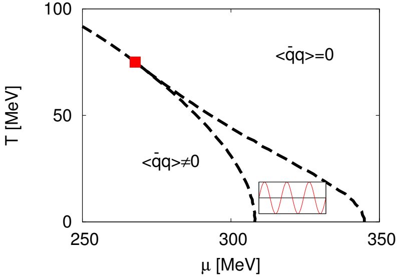
- Space dependent order parameter
- Popular for some time
  - Pion Condensation
  - (Color-) superconductivity
- Studied more recently in lower dimensional models (1+1 D Gross-Neveu model)



Modulated Order Parameter

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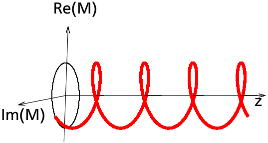
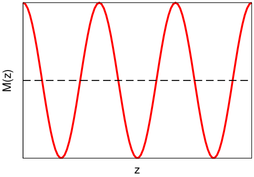
Different (periodical) modulations possible:

Solitonic modulation



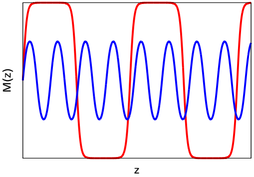
$$M(z) = \Delta\nu \; \mathrm{sn}(\Delta z|\nu)$$

Sinusoidal modulation



$$M(z) = M \; \cos(qz)$$

Chiral Density Wave



$$M(z) = M \; \exp(iqz)$$

NJL Model

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- NJL Lagrangian

$$\mathscr{L} = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi + G_S\left(\left(\overline{\psi}\psi\right)^2 + \left(\overline{\psi}i\gamma_5\tau^a\psi\right)^2\right)$$

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  - additional vector-interaction term

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- Derive thermodynamic properties from grand potential Ω

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  - additional vector-interaction term
- Derive thermodynamic properties from grand potential Ω
- Mean-field approximation

$$S(\vec{x}) = \langle\overline{\psi}\psi\rangle, \quad P(\vec{x}) = \langle\overline{\psi}i\gamma_5\tau^3\psi\rangle, \quad n^\mu(\vec{x}) = \langle\overline{\psi}\gamma^\mu\psi\rangle$$

  - keep space dependence, but neglect time dependence
  - consider only: $n^\mu(\vec{x}) = n(\vec{x})g^{\mu 0}$ (density)

Shifted Mass and Chemical Potential

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- Shifted mass and chemical potential

$$M(\vec{x}) = m - 2G_S \left(S(\vec{x}) + iP(\vec{x}) \right) , \qquad \tilde{\mu}(\vec{x}) = \mu - 2G_V n(\vec{x})$$

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- Hamiltonian $\mathscr{H} - \mu = \mathscr{H}_+ \otimes \mathscr{H}_-$

$$\mathscr{H}_+ = \begin{pmatrix} -i\vec{\sigma}\vec{\partial} - \tilde{\mu}(\vec{x}) & M(\vec{x}) \\ M^*(\vec{x}) & i\vec{\sigma}\vec{\partial} - \tilde{\mu}(\vec{x}) \end{pmatrix}$$

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- Demand periodicity in mass and chemical potential function

$$M(\vec{x}) = M(\vec{x} + \vec{n}_i), \quad \tilde{\mu}(\vec{x}) = \tilde{\mu}(\vec{x} + \vec{n}_i/2), \quad i = 1, 2, 3$$

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- Allows Fourier transformation

$$M(\vec{x}) = \sum_{\vec{q}_k} M_{\vec{q}_k} e^{i\vec{q}_k\vec{x}}, \qquad \tilde{\mu}(\vec{x}) = \sum_{\vec{q}_k} \tilde{\mu}_{\vec{q}_k} e^{i2\vec{q}_k\vec{x}}$$

- wave vector $\vec{q}_k$: $\quad \vec{q}_k\vec{n}_i = 2\pi N_{ki}, \quad N_{ki} \in \mathbb{Z}$

Grand Potential

Arrive at grand potential

$$\Omega = \Omega_{kin} + \Omega_{cond}$$

$$\Omega_{kin} = -N_C N_F \frac{1}{V} \sum_{E_\lambda} T \ln \left[2 \cosh \left(\frac{E_\lambda}{2T} \right) \right]$$

$$\Omega_{cond} = \frac{1}{V} \int d^3x \left[\frac{|M(\vec{x}) - m|^2}{4G_S} - \frac{(\tilde{\mu}(\vec{x}) - \mu)^2}{4G_V} \right]$$

with eigenvalues E_λ of $\mathscr{H}$ in momentum space

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$$\mathscr{H}_{\vec{p}_m,\vec{p}_{m'}} = \begin{pmatrix} -\vec{\sigma}\vec{p}_m \delta_{\vec{p}_m,\vec{p}_{m'}} + \sum_{\vec{q}_k} \tilde{\mu}_{\vec{q}_k} \delta_{2\vec{q}_k,(\vec{p}_m - \vec{p}_{m'})} & -\sum_{\vec{q}_k} M_{\vec{q}_k} \delta_{\vec{q}_k,(\vec{p}_m - \vec{p}_{m'})} \\ -\sum_{\vec{q}_k} M_{\vec{q}_k} \delta_{\vec{q}_k,(\vec{p}_{m'} - \vec{p}_m)} & \vec{\sigma}\vec{p}_m \delta_{\vec{p}_m,\vec{p}_{m'}} + \sum_{\vec{q}_k} \tilde{\mu}_{\vec{q}_k} \delta_{2\vec{q}_k,(\vec{p}_m - \vec{p}_{m'})} \end{pmatrix}$$

Simplify Hamiltonian

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- 1D dimensional simple real ansatz: Cosine

$$M(z) = M\cos(qz), \quad \tilde{\mu}(z) = \tilde{\mu}_0 + \tilde{\mu}_1 \cos(2qz)$$

- Use crystal properties
  - Momenta $\vec{p}$ and $\vec{p}'$ only coupled if they differ by integer multiples of q
  - construct momenta in modulated direction from reciprocal lattice vector q and vector in the first Brillouin zone k

$$p = k + mq, \qquad m \in \mathbb{Z}$$

  - momenta p and p' can only be coupled if $k = k'$
- cut matrix at high momenta

$$|k \pm mq| \leq \Lambda_M$$

Regularization and Determination of Parameters

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- kinetic part of grand potential divergent
- apply regularization scheme (Pauli-Villars)
- tune cutoff parameter Λ and coupling constant G_S to empiric values with vanishing bare quark mass $m = 0$ (chiral limit),
- treat vector coupling G_V as free parameter
- grand potential depends on 4 parameters M, q, $\tilde{\mu}_0$ and $\tilde{\mu}_1$
- parameters have to fulfill gap equations

$$\frac{\partial \Omega}{\partial M} = 0, \quad \frac{\partial \Omega}{\partial q} = 0, \quad \frac{\partial \Omega}{\partial \tilde{\mu}_0} = 0, \quad \frac{\partial \Omega}{\partial \tilde{\mu}_1} = 0$$

Effects of Vector Interactions on the Homogeneous Phase Diagram

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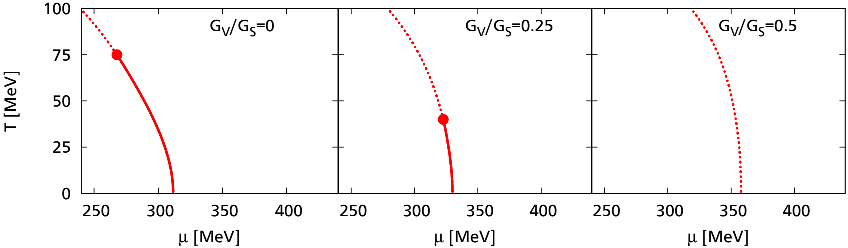

- First-order phase transition for vanishing vector coupling
- Rising vector coupling shifts critical end point to lower temperatures
- Hits the zero temperature axis for higher coupling constants
- Phase transition shifted to higher chemical potentials

Mass Amplitude and Wave Number

T = 1 MeV

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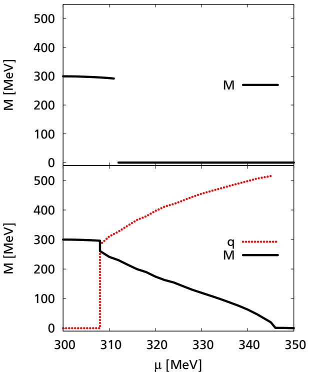
$G_V / G_S = 0$

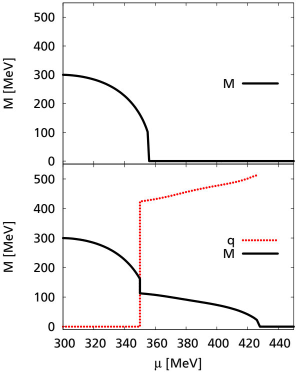


- First-order transition for homogeneous case
- Replaced by inhomogeneous region
- Small first-order transition from homogeneous broken to inhomogeneous phase
- Continous decrease in mass amplitude until chiral symmetry is restored

Mass Amplitude and Wave Number

$T = 1$ MeV

$G_V / G_S = 0.5$



- Second-order transition for homogeneous case
- Replaced by inhomogeneous phase
- Density

$$n(z) = \frac{\mu - \tilde{\mu}_0}{2G_V} - \frac{\tilde{\mu}_1}{2G_V}\cos(2qz) = \langle n \rangle - n_A \cos(2qz)$$

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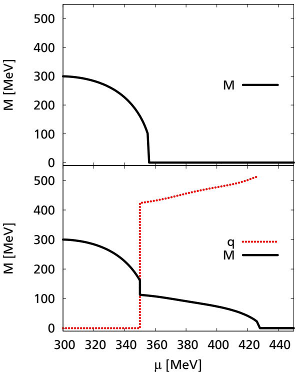

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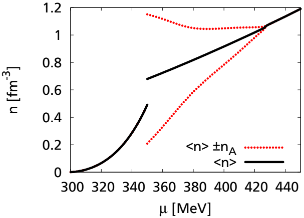

Different Modulations

Carignano, Nickel, Buballa (Phys. Rev. D 2010)

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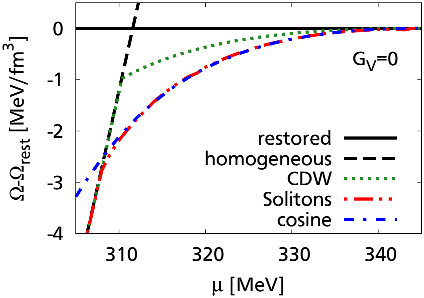
Solitonic Modulations

$$M(z) = \Delta\nu\,\mathrm{sn}(\Delta z|\nu)$$

Chiral density wave (CDW)

$$M(z) = M\exp(iqz)$$

- For both modulations analytic expression for eigenvalues are known



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- For both modulations analytic expression for eigenvalues are known
- Both can be extended to vector interactions by restricting to

$$n(\vec{x}) \to \overline{n} = \langle n(\vec{x})\rangle = \text{const.}$$
$$\Rightarrow \tilde{\mu} = \mu - 2G_V\overline{n} = \text{const.}$$

- Approximation for solitons, but exact for CDW
- Grand Potential

$$\Omega(T,\mu) = \Omega(T,\mu \to \tilde{\mu})|_{G_V=0} - \frac{(\tilde{\mu}-\mu)^2}{4G_V}$$

Comparison of Phase Diagrams and Free Energies at $G_V/G_S = 0.5$

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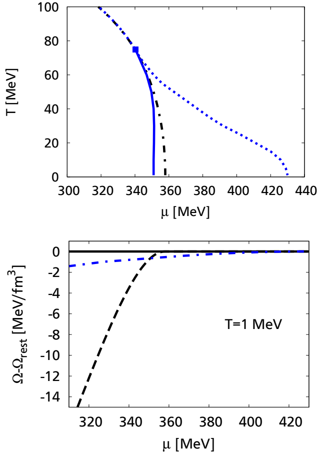

Compare the different modulations:

Sinusoidal: $M(z) = M\cos(qz)$

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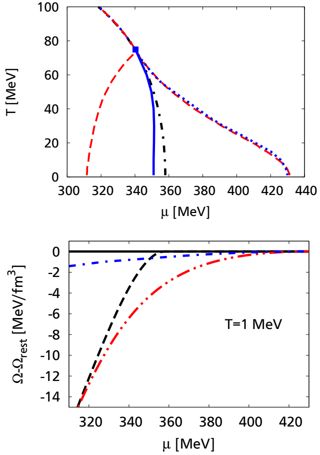

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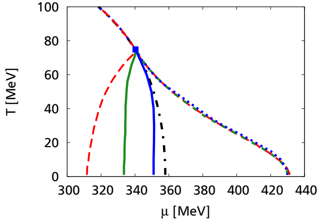

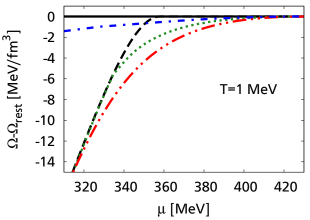

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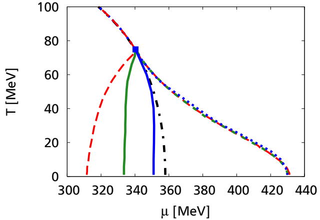

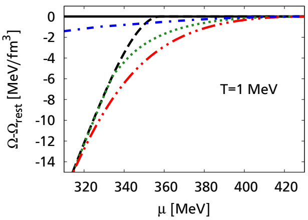

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Sinusoidal: $M(z) = M\cos(qz)$

Solitons: $M(z) = \Delta\nu\ \mathrm{sn}(\Delta z|\nu)$

CDW: $M(z) = M\exp(iqz)$

- Transition to restored phase similar for all modulations
- Solitons most favored but lack self consistency
- CDW favored over sinusoidal modulations
- Not the case without vector interactions

Comparison of Phase Diagrams and Free Energies at $G_V/G_S = 0.5$

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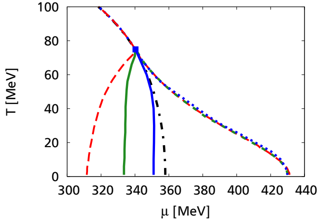

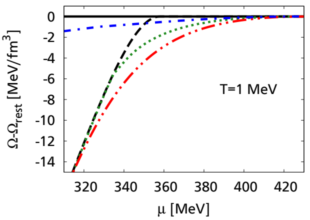

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$\Rightarrow$ Compare CDW and cosine for different G_V

Competition between CDW and Cosine

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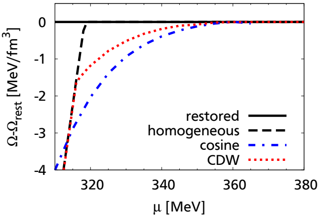
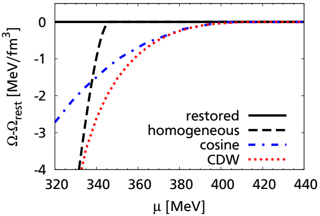
$G_V/G_S = 0.1$



- Sinusoidal modulation only favored for smaller vector couplings $G_V/G_S < 0.3$

Competition between CDW and Cosine

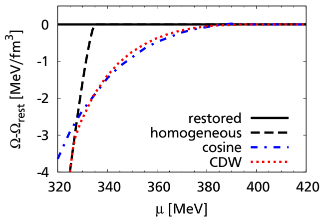
$G_V / G_S = 0.4$

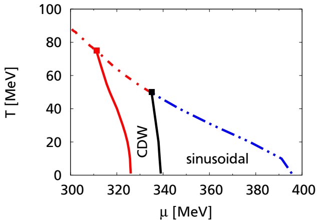


- Sinusoidal modulation only favored for smaller vector couplings $G_V / G_S < 0.3$
- CDW favored completely for $G_V / G_S \geq 0.4$

Competition between CDW and Cosine

$G_V/G_S = 0.3$





- Sinusoidal modulation only favored for smaller vector couplings $G_V/G_S < 0.3$
- CDW favored completely for $G_V/G_S \geq 0.4$
- Mixed inhomogeneous phase for $G_V/G_S = 0.3$

Conclusion and Outlook

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- Crystalline phase should be considered
- Inhomogeneous phases can be found with vector interactions
- Explicit modulations of the chemical potential show differences to constant $\tilde{\mu}$
- Sinusoidal modulations only favored at small vector couplings

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Outlook

- Different modulations, e.g. $M(z) = \Delta\nu \, \mathrm{sn}(\Delta z | \nu)$ with known Fourier decomposition
- 't Hooft interactions
- Isospin imbalance
  $\Rightarrow$ D. Nowakowski's talk tomorrow
- Phononic excitations in inhomogeneous phase