Local thermodynamical equilibrium and the hydrodynamic β frame

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Outline

- Local equilibrium in quantum relativistic theory
- The β frame
- Landau vs β frame
- Corrections to stress-energy tensor

NED meeting 2014

Motivations: relativistic hydrodynamics of strongly interacting fluids (QGP, cold atoms...)

- Full quantum-relativistic treatment

$$T^{\mu\nu}(x) \equiv \mathrm{tr}\left(\hat{\rho}\,\hat{T}^{\mu\nu}(x)\right)_{\mathrm{ren.}}$$

- Covariance maintained throughout

- Kinetic-theory free

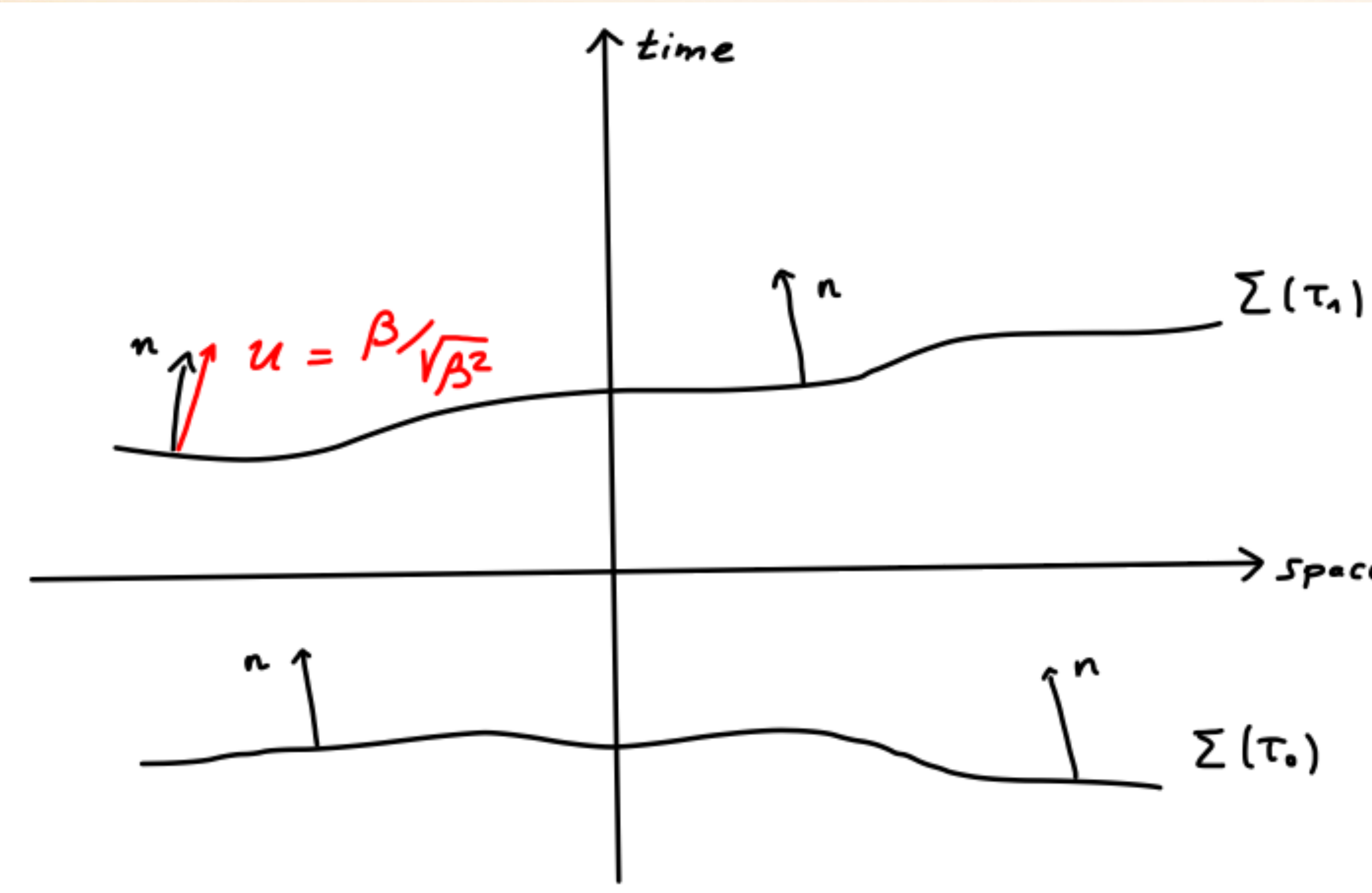
Local thermodynamical equilibrium

Maximization of entropy with fixed *densities* of conserved currents

$$S = -\mathrm{tr}(\hat{\rho}_{LE} \log \hat{\rho}_{LE}) \qquad \text{max. w.r.t. } \hat{\rho}_{LE}$$

$$n_\mu \mathrm{tr}(\widehat{\rho}_{\mathrm{LE}}\, \widehat{T}^{\mu\nu}(x))_{\mathrm{ren}} = n_\mu \langle \widehat{T}^{\mu\nu}(x) \rangle_{\mathrm{LE}} \equiv n_\mu T^{\mu\nu}_{\mathrm{LE}}(x) = n_\mu T^{\mu\nu}(x)$$
$$n_\mu \mathrm{tr}(\widehat{\rho}_{\mathrm{LE}} \widehat{j}^\mu(x))_{\mathrm{ren}} = n_\mu \langle \widehat{j}^\mu(x) \rangle_{\mathrm{LE}} \equiv n_\mu j^\mu_{\mathrm{LE}}(x) = n_\mu j^\mu(x)$$

As entropy is a non-conserved global quantity in LTE, it requires the specification of one frame, namely a 3d spacelike hypersurface with its normal *n* at any time (a spacetime foliation)



The vector field *n(x)* must be vorticity-free:

$$\epsilon_{\mu\nu\rho\sigma} n^\nu (\partial^\rho n^\sigma - \partial^\sigma n^\rho) = 0$$

Maximize:

$$-\mathrm{tr}(\widehat{\rho}\log\widehat{\rho}) + \int_{\Sigma(\tau)} \mathrm{d}\Sigma\, n_\mu \left[\left(\langle \widehat{T}^{\mu\nu}(x) \rangle - T^{\mu\nu}(x) \right) \beta_\nu(x) - \left(\langle \widehat{j}^\mu(x) \rangle - j^\mu(x) \right) \xi(x) \right]$$

The 5 Lagrange multiplier functions $\beta(x)$ and $\xi(x)$ become the primordial thermodynamic fields

$$\beta^\mu(x) = \frac{1}{T} u^\mu(x)$$

$$\xi(x) = \frac{\mu(x)}{T(x)}$$

The solution

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z_{\rm LE}} \exp\left[- \int_{\Sigma(\tau)} {\rm d}\Sigma \, n_\mu \left(\widehat{T}^{\mu\nu}(x)\beta_\nu(x) - \xi(x)\widehat{j}^\mu(x) \right) \right]$$

$$n_\mu T^{\mu\nu}_{\rm LE}[\beta, \xi, n] = n_\mu T^{\mu\nu} \qquad n_\mu j^\mu_{\rm LE}[\beta, \xi, n] = n_\mu j^\mu,$$

This operator, as expected, depends on the hypersurface Σ
It is independent if

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0 \qquad \partial_\mu \xi = 0.$$

β Killing vector field

Global equilibrium

$$\beta_\mu = b_\mu + \omega_{\mu\nu} x^\nu \qquad \zeta = \text{const.}$$

Reproduces all known forms of global equilibrium

$$\begin{cases} b = (1/T, \underline{0}) \\ \omega = 0 \end{cases} \rightarrow e^{-\hat{H}/T}$$

$$\begin{cases} b = \text{cost.} \\ \omega = 0 \end{cases} \rightarrow e^{-b\cdot\hat{P}}$$

$$\begin{cases} b = (1/T, \underline{0}) \\ \omega = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \underline{\omega} \end{array}\right) \end{cases} \rightarrow e^{-\hat{H}/T + \underline{\omega}\cdot\hat{J}/T}$$

If $\beta^\mu = b^\mu$ const.

$$\langle \hat{T}^{\mu\nu} \rangle_{eq} = (\rho + p) \frac{\beta^\mu \beta^\nu}{\beta^2} - p g^{\mu\nu}$$

LOCAL equilibrium depends on Σ

A choice has to be made. The simplest is to define LTE on the hypersurface orthogonal to β itself

$$n = \text{vers}\,\beta$$

The β frame

$$\beta_\mu T^{\mu\nu}_{\rm LE}[\beta,\xi] = \beta_\mu T^{\mu\nu} \qquad \beta_\mu j^\mu_{\rm LE}[\beta,\xi] = \beta_\mu j^\mu,$$

Equations defining β and ξ

This is possible if β is vorticity-free. If it is not, the definition has to be modified, but it can still be done

Features of the β frame

- Simplest form of entropy density $S = -\mathrm{tr}(\hat{\rho}_{LE} \log \hat{\rho}_{LE})$

$$\log Z = \int_\Sigma \mathrm{d}\Sigma_\mu \, \Phi^\mu \quad \Rightarrow \quad s^\mu = \phi^\mu + T^{\mu\nu}_{\mathrm{LE}}\beta_\nu - \xi j^\mu_{\mathrm{LE}} + s^\mu_T(n)$$

$$s^\mu n_\mu = n_\mu \phi^\mu + n_\mu T^{\mu\nu}\beta_\nu - \xi n_\mu j^\mu.$$

$$\sqrt{\beta^2} s = \beta \cdot \phi + \beta_\mu \beta_\nu T^{\mu\nu} - \xi \beta_\mu j^\mu,$$

- *vers* β is the four-velocity of an ideal relativistic thermometer, that is an object able to reach equilibrium with respect to energy and momentum echange with the fluid in x

Relativistic hydro in the β frame

Equation of state

$p(\beta^2, \xi)$

Ideal s.e.t.

$$T^{\mu\nu}_{\rm id} = -2\frac{\partial p}{\partial \beta^2}\beta^\mu\beta^\nu - pg^{\mu\nu},$$

$$-2\frac{\partial p}{\partial \beta^2} = \frac{\rho + p}{\beta^2} = \frac{h}{\beta^2}.$$

All gradients in u, T can be recasted as gradients of β

$$\sqrt{\beta^2}\Delta_{\mu\nu}D\beta^\nu + \frac{1}{2}\nabla_\mu\beta^2 = \beta_\lambda\Delta_{\mu\nu}(\partial^\lambda\beta^\nu + \partial^\nu\beta^\lambda)$$

$$\sqrt{\beta^2}\Delta_{\mu\nu}D\beta^\nu + \frac{1}{2}\nabla_\mu\beta^2 = \beta^2 A^\mu + \frac{1}{2}\nabla_\mu\frac{1}{T^2} = \frac{1}{T^2}\left(A_\mu - \frac{1}{T}\nabla_\mu T\right).$$

$$\nabla_\mu u^\nu = \nabla_\mu\frac{\beta^\nu}{\sqrt{\beta^2}} = \beta^\nu\left(-\frac{1}{2}\right)(\beta^2)^{-3/2}\nabla_\mu\beta^2 + \frac{1}{\sqrt{\beta^2}}\nabla_\mu\beta^\nu$$
$$= \frac{1}{\sqrt{\beta^2}}\left(-\frac{\beta^\nu\beta^\rho}{\beta^2}\nabla_\mu\beta_\rho + \nabla_\mu\beta^\nu\right) = \frac{1}{\sqrt{\beta^2}}\Delta^{\rho\nu}\nabla_\mu\beta_\rho,$$

$$\nabla_\mu u^\mu = \frac{1}{\sqrt{\beta^2}}\nabla_\mu\beta^\mu.$$

Actual vs LTE density operator (see E. Grossi's talk)

$$\partial_\mu T^{\mu\nu} = \partial_\mu \mathrm{tr}(\widehat{\rho}\widehat{T}^{\mu\nu})_{\mathrm{ren}} = \mathrm{tr}(\widehat{\rho}\,\partial_\mu \widehat{T}^{\mu\nu})_{\mathrm{ren}} = 0,$$

where **ρ** is the fixed density operator in the Heisenberg picture.
In hydro, it is the initial LTE operator, as used in Zubarev method

$$\widehat{\rho} \equiv \widehat{\rho}_{LE}(\tau_0) = \frac{1}{Z_{LE}(\tau_0)} \exp\left[- \int_{\tau_0} d\Sigma\, n_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \xi \widehat{j}^\mu \right)\right]$$

The true mean value of an operator can be expanded from present LTE

$$\langle \widehat{T}^{\mu\nu}(x) \rangle \simeq \langle \widehat{T}^{\mu\nu}(x) \rangle_{LE} + O(\partial\beta, \partial\xi)$$

$\eta, \zeta, \dots$

Landau vs β frame

Eckart $j^\mu = n\, u^\mu_E$

Landau $T^{\mu\nu} u_{\nu_L} = \lambda\, u^\mu_L$

Beta $\beta_\mu T^{\mu\nu}_{\rm LE}[\beta,\xi] = \beta_\mu T^{\mu\nu} \qquad \beta_\mu j^\mu_{\rm LE}[\beta,\xi] = \beta_\mu j^\mu,$

Landau vs β frame (cont'd)

If $\langle \hat{T}^{\mu\nu}_{LE} \rangle = \langle \hat{T}^{\mu\nu}_{id} \rangle$ they coincide

Yet, in general

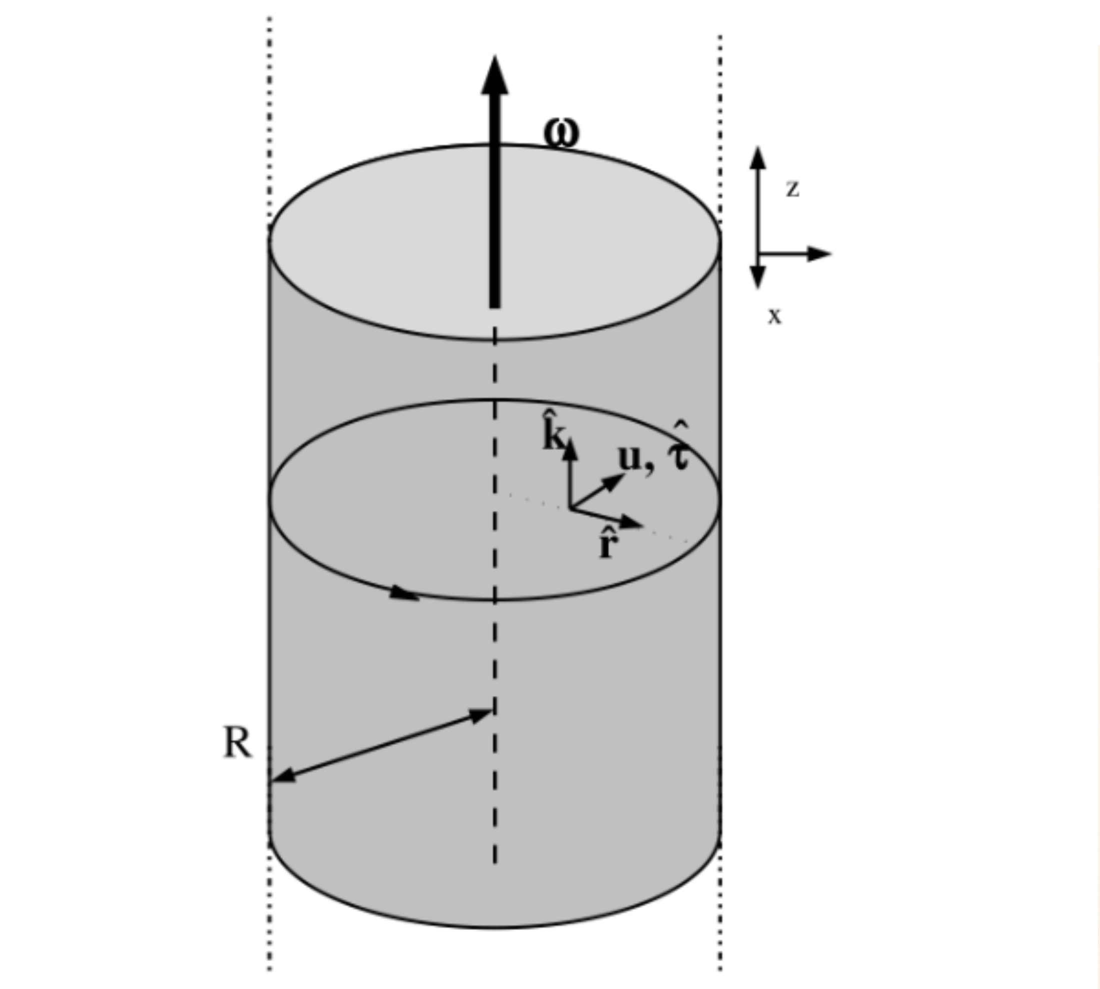
$$T^{\mu\nu}_{\rm LE}(x) = {\rm tr}(\widehat{\rho}_{\rm LE}\widehat{T}^{\mu\nu}(x))_{\rm ren} = \frac{1}{Z_{\rm LE}}{\rm tr}\left(\exp\left[-\int {\rm d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \xi\widehat{j}^\mu\right)\right]\widehat{T}^{\mu\nu}(x)\right)_{\rm ren}$$

Only if β and ξ are constant

$$\frac{1}{Z}{\rm tr}\left(e^{-\beta\cdot\hat{P}+\xi\hat{Q}}\hat{T}^{\mu\nu}(x)\right) = T^{\mu\nu}(x)_{id} = (\rho+p)\frac{\beta^\mu\beta^\nu}{\beta^2} - p g^{\mu\nu}$$

β # Landau. An equilibrium calculation: the free scalar field with rotation

$$\widehat{T}^{\mu\nu} = \partial^{(\mu}\widehat{\psi}\,\partial^{\nu)}\widehat{\psi} - g^{\mu\nu}\widehat{\mathcal{L}} \qquad \widehat{\mathcal{L}} = \frac{1}{2}\left(\partial_\mu\widehat{\psi}\,\partial^\mu\widehat{\psi} - m^2\widehat{\psi}^2\right)$$



$$\widehat{\rho} = \frac{1}{Z}\exp[-\widehat{H}/T + \omega\cdot\widehat{J}_z/T]\mathsf{P}_V,$$

In this case $\beta = (1/T)(1, \boldsymbol{\omega}\times\mathbf{x})$

Keeping in mind $\beta_\mu = b_\mu + \omega_{\mu\nu}x^\nu$

$$\varpi_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu\right)$$

$$\omega_{\lambda\nu} = \omega/T\left(\delta^1_\lambda\delta^2_\nu - \delta^2_\lambda\delta^1_\nu\right)$$

It can be shown by explicit calculation

$$
T = \begin{pmatrix} u\cdot T\cdot u & u\cdot T\cdot \hat{\tau} & 0 & 0 \\ u\cdot T\cdot \hat{\tau} & \hat{\tau}\cdot T\cdot \hat{\tau} & 0 & 0 \\ 0 & 0 & \hat{r}\cdot T\cdot \hat{r} & 0 \\ 0 & 0 & 0 & k\cdot T\cdot k \end{pmatrix}
$$

$$
\hat{r}\cdot T\cdot \hat{r} \neq \hat{k}\cdot T\cdot \hat{k}
$$

$$
\hat{r}\cdot T\cdot \hat{r} - k\cdot T\cdot k = \sum_{M=-\infty}^{+\infty}\sum_{p_T}\int \mathrm{d}p_L \, \frac{2}{(2\pi)^2\,\varepsilon\,R^2\,J_M'^2(p_T R)}\,\frac{1}{\mathrm{e}^{\beta(\varepsilon - M\omega)}-1}\left[p_T^2 J_M'(p_T r)^2 - p_L^2 J_M(p_T r)^2\right],
$$

$$
\langle \hat{T}^{\mu\nu}\rangle_{eq} \neq (\rho + p)\, u^\mu u^\nu - p\, g^{\mu\nu}
$$

$$
\langle \hat{T}^{\mu\nu}\rangle_{eq} = -\frac{\partial p}{\partial \beta^2}\,\beta^\mu\beta^\nu - p(\beta^2,\xi)\,g^{\mu\nu} + O\left[\left(\frac{\hbar\omega}{KT}\right)^2\right]
$$

Local equilibrium expansion

$$T^{\mu\nu}_{\rm LE}(x) = {\rm tr}(\widehat{\rho}_{\rm LE}\widehat{T}^{\mu\nu}(x))_{\rm ren} = \frac{1}{Z_{\rm LE}}{\rm tr}\left(\exp\left[-\int {\rm d}\Sigma_\mu\ \left(\widehat{T}^{\mu\nu}\beta_\nu - \xi\widehat{j}^\mu\right)\right]\widehat{T}^{\mu\nu}(x)\right)_{\rm ren}$$

The correlation length between the stress-energy tensor in hydrodynamic limit is much smaller than the length of β variation, so the exponent can be expanded in a Taylor series from x

$$\exp\left[-\int {\rm d}\Sigma_\mu\ \left(\widehat{T}^{\mu\nu}\beta_\nu - \xi\widehat{j}^\mu\right)\right]$$
$$\simeq \exp\left[-\beta_\nu(x)\widehat{P}^\nu + \xi(x)\widehat{Q} - \nabla_\lambda\beta_\nu(x)\int_{T\Sigma}{\rm d}\Sigma_\mu(y)\,\widehat{T}^{\mu\nu}(y)(y^\lambda - x^\lambda) + \nabla_\lambda\xi(x)\int_{T\Sigma}{\rm d}\Sigma_\mu(y)\,\widehat{j}^\mu(y)(y^\lambda - x^\lambda) + \ldots\right]$$

$$T^{\mu\nu}_{\rm LE}(x) \simeq \frac{1}{Z_{\rm eq}(\beta(x),\xi(x))}{\rm tr}\left(\exp\left[-\beta_\nu(x)\widehat{P}^\nu + \xi(x)\widehat{Q}\right]\widehat{T}^{\mu\nu}(x)\right)_{\rm ren} + \mathcal{O}(\partial\beta,\partial\xi).$$

$$T^{\mu\nu}_{\rm LE}(x) \simeq T^{\mu\nu}_{\rm id}(x) + \mathcal{O}(\nabla\beta,\nabla\xi) = (\rho+p)_{\rm eq}\frac{1}{\beta^2}\beta^\mu(x)\beta^\nu(x) - p_{\rm eq}g^{\mu\nu} + \mathcal{O}(\partial\beta,\partial\xi),$$

Vorticous case

Define the field *b(x)* inspired by the equilibrium solution

$$\beta_\nu(x) \equiv b_\nu(x) + \varpi_{\nu\lambda}(x) x^\lambda,$$

$$\varpi_{\nu\lambda}(x) = -\frac{1}{2}(\partial_\nu \beta_\lambda - \partial_\lambda \beta_\nu) - \frac{1}{6}\left(x^\rho \partial_\rho \partial_\nu \beta_\lambda - x^\rho \partial_\rho \partial_\lambda \beta_\nu\right) + \dots$$

This is not vorticous and can be taken as frame field

$$(\beta_\mu - \varpi_{\mu\lambda} x^\lambda)\delta T^{\mu\nu} = 0 \qquad (\beta_\mu - \varpi_{\mu\lambda} x^\lambda)\delta j^\mu = 0$$

$$\delta T^{\mu\nu} = T^{\mu\nu} - T^{\mu\nu}_{LE}$$

Take $\xi = 0$ for simplicity and expand the exponent of the LTE operator

$$
\begin{aligned}
-\int \mathrm{d}\Sigma\, n_\mu \widehat{T}^{\mu\nu}\beta_\nu &= -\int \mathrm{d}\Sigma(y)\, n_\mu \widehat{T}^{\mu\nu}(b_\nu + \varpi_{\nu\lambda} y^\lambda) \\
&= -\int \mathrm{d}\Sigma(y)\, n_\mu \widehat{T}^{\mu\nu} b_\nu - \frac{1}{2}\varpi_{\lambda\nu}(y^\lambda \widehat{T}^{\mu\nu} - y^\nu \widehat{T}^{\mu\lambda})
\end{aligned}
$$

$$
\begin{aligned}
&-\int \mathrm{d}\Sigma(y)\, n_\mu \left[\widehat{T}^{\mu\nu} b_\nu - \frac{1}{2}\varpi_{\lambda\nu}(y^\lambda \widehat{T}^{\mu\nu} - y^\nu \widehat{T}^{\mu\lambda})\right] \\
\simeq& -b_\nu(x)\int \mathrm{d}\Sigma(y)\, n_\mu \widehat{T}^{\mu\nu} - \frac{\partial b_\nu}{\partial x^\rho}\int \mathrm{d}\Sigma(y)\, n_\mu (y^\rho - x^\rho)_T \widehat{T}^{\mu\nu} + \frac{1}{2}\varpi_{\lambda\nu}(x)\int \mathrm{d}\Sigma(y)\, n_\mu (y^\lambda \widehat{T}^{\mu\nu} - y^\nu \widehat{T}^{\mu\lambda}) \\
=& -b_\nu(x)\widehat{P}^\nu - \frac{\partial b_\nu}{\partial x^\rho}\int \mathrm{d}\Sigma(y)\, n_\mu (y^\rho - x^\rho)_T \widehat{T}^{\mu\nu} + \frac{1}{2}\varpi_{\lambda\nu}(x)\widehat{J}^{\lambda\nu} \\
=& -b_\nu(x)\widehat{P}^\nu - \frac{1}{4}(\partial_\rho b_\nu + \partial_\nu b_\rho)\int \mathrm{d}\Sigma(y)\, n_\mu \left[(y^\rho - x^\rho)_T \widehat{T}^{\mu\nu} + (y^\nu - x^\nu)_T \widehat{T}^{\mu\rho}\right] + \frac{1}{2}\varpi_{\lambda\nu}(x)\widehat{J}^{\lambda\nu}
\end{aligned}
$$

Result

$$-b_\nu(x)\widehat{P}^\nu + \frac{1}{2}\varpi_{\lambda\nu}(x)\widehat{J}^{\lambda\nu} - \frac{1}{2}(\nabla_\lambda\beta_\nu + \nabla_\nu\beta_\lambda)\widehat{L}_x^{\lambda\nu}$$

$$\widehat{L}_x^{\lambda\nu} \equiv \frac{1}{2}\int_{T\Sigma} \mathrm{d}\Sigma(y)\, n_\mu(y^\lambda - x^\lambda)\,\widehat{T}^{\mu\nu}(y) + (\lambda \leftrightarrow \nu)$$

Taking into account that

$$\widehat{J}^{\lambda\nu} = \widehat{J}_x^{\lambda\nu} + x^\lambda \widehat{P}^\nu - x^\nu \widehat{P}^\lambda$$

the LTE exponent can be finally written

$$-\beta_\nu(x)\widehat{P}^\nu + \frac{1}{2}\varpi_{\lambda\nu}(x)\widehat{J}_x^{\lambda\nu} - \frac{1}{2}(\nabla_\lambda\beta_\nu + \nabla_\nu\beta_\lambda)\widehat{L}_x^{\lambda\nu}$$

Local velocity — Local vorticity — Local shear

Outlook and conclusions

- The β frame is a well suited frame for strongly interacting fluids, when a kinetic description is impossible
- It allows to treat hydrodynamics in a quantum relativistic framework and to properly express the condition of local thermodynamical equilibrium and to expand from it (Zubarev approach, see talk by E. Grossi)
- β does not coincide with Landau whenever the stress-energy tensor at LTE differs from the ideal form
- Stress energy tensor at LTE gets 2nd order quantum corrections depending on local vorticity (and acceleration). Are they relevant for hydro applied to QGP?

Polarization

The β field enters in the expression of the polarization vector of particles in a (thermal)vorticous motion

$$\langle \Pi_\mu(x,p) \rangle \simeq \frac{1}{8} \epsilon_{\mu\rho\sigma\tau} \, (1-n_F) \partial^\rho \beta^\sigma \frac{p^\tau}{m}$$

F.B., V. Chandra, L. Del Zanna, E. Grossi, Ann. Phys. 338 (2013) 32
F.B., L. Csernai, D.J. Wang, Phys. Rev. C 88 (2013) 034905