Zubarev nonequilibrium density operator and transport coefficients

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Outline

- Zubarev non equilibrium density matrix
- Local thermodynamical equilibrium terms
- Second order terms in the gradient expansion

Zubarev non equilibrium density matrix

D. N. Zubarev and M. V. Tokarchuk, Theor. Math. Phys. **88**, 876 (1992).

$$\partial_\mu T^{\mu\nu}(x) = \partial_\mu \mathrm{tr}(\widehat{\rho}\widehat{T}^{\mu\nu}) = \mathrm{tr}(\widehat{\rho}\partial_\mu \widehat{T}^{\mu\nu}) = 0$$

ρ density operator in Heisenberg picture

The hydrodynamical problem is to determine the evolution of the mean values starting from LTE (local thermodinamical equilibrium) condition at some time τ_0

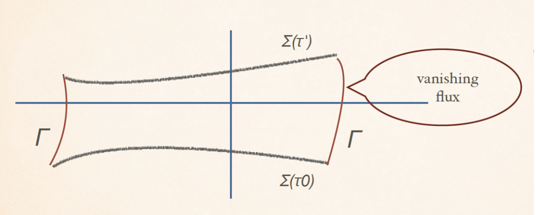
$$\widehat{\rho}_{LE} = \frac{1}{Z_{\mathrm{LE}}(\tau_0)} \exp\left[-\int_{\Sigma(\tau_0)} \mathrm{d}\Sigma\, n_\mu \left(\widehat{T}^{\mu\nu}(x)\beta_\nu(x) - \xi(x)\widehat{j}^\mu(x)\right)\right]$$

- Σ 3d spacelike hyper-surface with normal vector n^μ
- $\beta(x)$ local four-temperature
- $\xi(x) = \mu(x)/T(x)$ local chemical potential

Non equilibrium density matrix

The stationary density matrix can be transformed using the Gauss theorem

$$-\int_{\Sigma(\tau_0)} \mathrm{d}\Sigma \; n_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \widehat{j}^\mu \xi \right) = -\int_{\Sigma(\tau')} \mathrm{d}\Sigma \; n_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \widehat{j}^\mu \xi \right) + \int_\Omega \mathrm{d}\Omega \; \left(\widehat{T}^{\mu\nu} d_\mu \beta_\nu - \widehat{j}^\mu d_\mu \xi \right),$$



$$\widehat{\rho} = \frac{1}{Z} \exp \left[-\int_{\Sigma(\tau')} \mathrm{d}\Sigma \; n_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \widehat{j}^\mu \xi \right) + \int_\Omega \mathrm{d}\Omega \; \left(\widehat{T}^{\mu\nu} d_\mu \beta_\nu - \widehat{j}^\mu d_\mu \xi \right) \right]$$

If the system stays close to LTE at any time, the second term is a correction and an expansion can be made from present LTE

Non equilibrium density matrix

The non equilibrium density matrix includes two terms
$\rho = e^{A+B}/Z$

- $A = -\int_{\Sigma(\tau')} \mathrm{d}\Sigma \, n_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^\mu \xi \right)$ leading term local equilibrium
- $B = \int_\Omega \mathrm{d}\Omega \, \left(\widehat{T}^{\mu\nu} d_\mu \beta_\nu - \widehat{j}^\mu d_\mu \xi \right)$ dissipative term

We can consider the dissipative term as a perturbation and expand the mean value of an operator (linear response):

$$\langle \widehat{O}(x') \rangle \simeq \langle \widehat{O}(x') \rangle_{\mathrm{LE}} - \langle \widehat{O}(x') \rangle_{\mathrm{LE}} \langle \widehat{B} \rangle_{\mathrm{LE}} + \int_0^1 \mathrm{d}z \, \langle \widehat{O}(x') \mathrm{e}^{z\widehat{A}} \widehat{B} \mathrm{e}^{-z\widehat{A}} \rangle_{\mathrm{LE}}$$

Linear Response theory

After standard manipulations

$$\langle O(x')\rangle - \langle O(x')\rangle_{LE} \simeq iT \int_{\tau_0}^{\tau'} \mathrm{d}^4x \int_{\tau_0}^{\tau} \mathrm{d}\theta \left(\langle [\widehat{O}(x'), \widehat{T}^{\mu\nu}(\theta, \mathbf{x})] \rangle_{\beta(x')} \partial_\mu \beta_\nu(x) \right)$$

Define $\delta\beta = \beta(x) - \beta_{eq}$ in the hydro limit, this perturbation varies on a scale much larger than the operators correlation length, thus only the smallest Fourier component can be retained

$$\delta\beta_\nu(x) \simeq A_\nu \frac{1}{2i} (\mathrm{e}^{-iK\cdot(x-x')} - \mathrm{e}^{iK\cdot(x-x')})$$

and after some manipulation..

$$\delta\langle\widehat{O}(x')\rangle = \partial_\mu\beta_\nu(x') \frac{\mathrm{d}}{\mathrm{d}\omega}\Big|_{\omega=0} \lim_{\mathbf{k}\to\mathbf{0}} \mathrm{Im}\, iT \int_{-\infty}^{0} \mathrm{d}^4y\, \langle [\widehat{O}(x'), \widehat{T}^{\mu\nu}(y)] \rangle_T \mathrm{e}^{-ik\cdot y}$$

the well known Kubo formula for the first order transport coefficients.

A. Hosoya, M. -a. Sakagami and M. Takao, Annals Phys. **154**, 229 (1984).

Local equilibrium mean values

To compute the LTE mean value for an operator at some point $\langle O(x')\rangle_{LE}$

$$\langle \widehat{O}(x')\rangle_{LE} = \frac{1}{Z_{LE}} \mathrm{tr}\left[\widehat{O}(x') e^{-\int_{\Sigma(\tau')} \mathrm{d}\Sigma \, n_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^\mu \xi \right)} \right]$$

The mean value at LE can be approximated Taylor expanding $\beta(x)$ and $\xi(x)$ from the point x' where the operator is evaluated .

$$-\int_{\Sigma(\tau')} \mathrm{d}\Sigma \, n_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^\mu \xi \right) \simeq -\beta_\nu(x')\widehat{P^\nu} + \xi(x')\widehat{Q} + \mathcal{O}(\partial)$$

then, at the zeroth order

$$\langle \widehat{O}(x')\rangle_{LE} \simeq \frac{1}{Z(\beta(x'))} \mathrm{tr}\left[\widehat{O}(x') e^{-\beta_\nu(x')\widehat{P}^\nu + \xi(x')\widehat{Q}} \right] = \langle \widehat{O}(x')\rangle_{\beta(x')}$$

Local equilibrium mean values: first order corrections

The density matrix for local equilibrium reads($\xi(x) = 0$ for simplicity):

$$\widehat{\rho} = \frac{1}{Z_{LE}} \exp[-\beta_\nu(x)\widehat{P}^\nu + \frac{1}{2}\omega_{\nu\lambda}(x)\widehat{J}_x^{\nu\lambda} - \frac{1}{2}(\nabla_\lambda\beta_\nu + \nabla_\nu\beta_\lambda)\widehat{L}_x^{\nu\lambda}]$$

- $\omega^{\nu\lambda}(x) = -\frac{1}{2}(\partial^\nu\beta^\lambda - \partial^\lambda\beta^\nu) - \cdots$ is the thermal vorticity
- $\widehat{J}_x^{\mu\lambda} = \int_\Sigma \mathrm{d}\Sigma(y)\, n_\mu \widehat{T}^{\mu\nu}(y)(y-x)^\lambda - (\nu \leftrightarrow \lambda)$
- $\widehat{L}_x^{\nu\lambda} = \int_\Sigma \mathrm{d}\Sigma(y)\, n_\mu \widehat{T}^{\mu\nu}(y)(y-x)^\lambda + (\nu \leftrightarrow \lambda)$

For the global equilibrium $\omega = cost$ reproduces the equilibrium with rotation

First order correction to stress-energy tensor

The first order corrections to the mean value of the stress energy tensor can be computed again using liner response theory

$$\langle \widehat{T}^{\mu\nu} \rangle_{LE} - \langle \widehat{T}^{\mu\nu} \rangle_{id} = -\frac{\partial_i \beta_\lambda(x)}{i\beta(x)} \int_{-\infty}^{0} \mathrm{d}^4 x x^i \langle [\widehat{T}^{0\lambda}(x), \widehat{T}^{\mu\nu}(0)] \rangle_T$$

in Fourier transform becomes:

$$\langle \widehat{T}^{\mu\nu} \rangle_{LE} - \langle \widehat{T}^{\mu\nu} \rangle_{id} \quad = \lim_{p \to 0} \frac{\partial_i \beta_\lambda}{\beta} \frac{\partial}{\partial p^i} \int_{-\infty}^{0} \mathrm{d}^4 x \langle [\widehat{T}^{\mu\nu}(0), \widehat{T}^{0\lambda}(x)] \rangle e^{-ip\cdot x}$$

These are the terms produced by the "shear" operator $\widehat{L}_x$ that vanish at global equilibrium.

First order correction to stress-energy tensor

$$\langle \widehat{T}^{\mu\nu} \rangle_{LE} - \langle \widehat{T}^{\mu\nu} \rangle_{id} = \partial_i \beta_\lambda \lim_{p_0, \mathbf{p} \to 0} \frac{\partial}{\partial p_i} \mathrm{Im} G_R(p_0, \mathbf{p})$$

Since $\mathrm{Im}G$ is an odd function of the energy p_0 because of PT symmetry

$$\mathrm{Im} G_R(p_0, \mathbf{p}) = -\mathrm{Im} G_R(-p_0, \mathbf{p})$$

then the first order contribution vanish

$$\langle \widehat{T}^{\mu\nu} \rangle_{LE} - \langle \widehat{T}^{\mu\nu} \rangle_{id} = \mathcal{O}(\partial^2)$$

Local equilibrium: second order corrections

$$\langle T^{\mu\nu}(x)\rangle_{LE} = \frac{1}{Z_{LE}}\mathrm{tr}(\widehat{T}^{\mu\nu}(x)e^{-\beta_\nu(x)\widehat{P}^\nu+\frac{1}{2}\omega_{\lambda\nu}(x)\widehat{J}_x^{\lambda\nu}-\frac{1}{2}(\nabla_\lambda\beta_\nu+\nabla_\nu\beta_\lambda)\widehat{L}_x^{\nu\lambda}})$$

It is possible to expand up to second order using twice the identity (Kubo-Identity):

$$e^{\widehat{A}+\widehat{B}} = e^{\widehat{A}}\left[1+\int_0^1 \mathrm{d}\lambda e^{-\lambda\widehat{A}}\widehat{B}e^{\lambda(\widehat{A}+\widehat{B})}\right]$$

This expansion yields the 3 point retarded function

$$\begin{aligned} G_{R1}^{ABC}(x,x_1,x_2) &= \theta(t-t_1)\theta(t_1-t_2)\langle[[\widehat{A}(x),\widehat{B}(x_1)],\widehat{C}(x_2)]\rangle_T \\ &+ \theta(t-t_2)\theta(t_2-t_1)\langle[[\widehat{A}(x),\widehat{C}(x_2)],\widehat{B}(x_1)]\rangle_T \end{aligned}$$

Second order local equilibrium terms

Local Vorticity contribution

$$\langle T^{\mu\nu}(x)\rangle_{LE} \simeq \frac{\omega_{\alpha\beta}(x)\omega_{\rho\sigma}(x)}{-8\beta^2}\int_{-\infty}^{t}\mathrm{d}^4x_1\int_{-\infty}^{t}\mathrm{d}^4x_2\theta(t-t_1)\theta(t_1-t_2)$$
$$\langle[[\widehat{T}^{\mu\nu}(x),(x_1-x)^{[\alpha}\widehat{T}^{0\beta]}(x_1)],(x_2-x)^{[\sigma}\widehat{T}^{0\rho]}(x_2)]\rangle_T$$
$$+((\alpha,\beta,x_1)\leftrightarrow(\sigma,\rho,x_2))$$
$$=\frac{\omega_{\alpha\beta}(x)\omega_{\rho\sigma}(x)}{-8\beta^2}\int_{-\infty}^{t}\mathrm{d}^4x_1\int_{-\infty}^{t}\mathrm{d}^4x_2G_{R1}^{T^{\mu\nu}J^{\alpha\beta}J^{\rho\sigma}}(x,x_1,x_2)$$

Local Shear contribution

$$\frac{\nabla_{(\alpha}\beta_{\lambda)}(x)\nabla_{(\rho}\beta_{\sigma)}(x)}{-8\beta^2}\int_{-\infty}^{t}\mathrm{d}^4x_1\int_{-\infty}^{t}\mathrm{d}^4x_2G_{R1}^{T^{\mu\nu}L^{\alpha\lambda}L^{\rho\sigma}}(x,x_1,x_2)$$

Comparison with other calculations

G. D. Moore and K. A. Sohrabi, Phys. Rev. Lett. **106**, 122302 (2011)

P. Arnold, D. Vaman, C. Wu and W. Xiao, JHEP **1110**, 033 (2011)

The second order Kubo formulae are obtained turning on a metric perturbation $g^{\mu\nu}(x) = \eta^{\mu\nu} + h^{\mu\nu}(x)$ and expanding in series the mean value of the stress tensor in power of $h^{\mu\nu}$

$$\langle \widehat{T}^{xy} \rangle = \int \mathrm{d}^4 x h_{xy} G_R^{xy,xy}(x) + \cdots$$

To find the transport coefficients we need to solve the hydro equation in curved background, then expanding in h

$$\nabla_\mu T^{\mu\nu} = 0 \to T^{xy} = -P h^{xy} - \eta \partial_t h^{xy} + \cdots$$

Matching the above expressions one identifies transport coefficients

Second order terms of the stress energy tensor

The second order expression for the stress energy tensor

R. Baier, P. Romatschke, D. T. Son, A. O. Starinets and M. A. Stephanov, JHEP **0804**, 100 (2008)

$$
\begin{aligned}
\eta\tau_\pi(u\cdot\nabla\sigma^{\mu\nu} + \frac{\nabla\cdot u}{3}\sigma^{\mu\nu}) + \kappa(R^{\langle\mu\nu\rangle} - 2u_\alpha u_\beta R^{\alpha\langle\mu\nu\rangle\beta}) \\
+\lambda_1\sigma_\lambda^{\langle\mu}\sigma_\lambda^{\mu\rangle\lambda} + \lambda_2\sigma_\lambda^{\langle\mu}\Omega^{\nu\rangle\lambda} + \lambda_3\Omega_\lambda^{\langle\mu}\Omega^{\nu\rangle\lambda} \\
+\eta\tau_\pi^*\frac{\nabla\cdot u}{3}\sigma^{\mu\nu} + \lambda_4\nabla^{\langle\mu}\ln s\nabla^{\nu\rangle}\ln s + \kappa^* 2u_\alpha u_\beta R^{\alpha\langle\mu\nu\rangle\beta} \\
+\Delta^{\mu\nu}(-\zeta\tau_\Pi u\cdot\nabla\nabla\cdot u + \xi_1\sigma^{\alpha\beta}\sigma_{\alpha\beta} + \xi_2(\nabla\cdot u)^2 \\
+\xi_3\Omega_{\alpha\beta}\Omega^{\alpha\beta} + \xi_4\nabla^\alpha\ln s\nabla^\beta\ln s + \xi_5 R + \xi_6 u^\alpha u^\beta R_{\alpha\alpha})
\end{aligned}
$$

- ∇ Covariant derivative
- $\Delta^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$ Transverse projector
- $\sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}(\nabla_\alpha u_\beta + \alpha \leftrightarrow \beta - \frac{2}{3}\Delta_{\alpha\beta}\nabla_\gamma u^\gamma))$
- $\Omega_{\mu\nu} = \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(\nabla_\alpha u_\beta - \nabla_\beta u_\alpha)$ vorticity
- $R^{\mu\nu\alpha\beta}$ Curvature tensor

Vorticity contribution

Ω is orthogonal to u, ω is not

$$\omega_{\mu\nu}u^{\nu} \neq 0 \qquad \Omega_{\mu\nu}u^{\nu} = 0$$

λ_3 coefficient is finite, non-vanishing, also at equilibrium.

$$\lambda_3 = -4 \lim_{p,q\to 0} \frac{\partial^2}{\partial p^z \partial q^z} \int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2 G_{R1}^{xy,x0,y0}(0,x_1,x_2) e^{-i(x_1\cdot p + x_2\cdot q)}$$

G. D. Moore and K. A. Sohrabi, JHEP 1211, 148 (2012)

We are comparing these expressions with those obtained in our approach and also evaluate the longitudinal contribution.

Summary and Outlook

- We are studying the local equilibrium correction to the stress energy tensor
- The most important correction seems to be quadratic in vorticity
- We are going to compare our approach with the other calculation.