Constraining viscous hydrodynamical evolution

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1. Abstract

We show that by requiring positivity of the longitudinal pressure it is possible to constrain the initial conditions one can use in 2nd-order viscous hydrodynamical simulations of ultrarelativistic heavy-ion collisions. We demonstrate this explicitly for 0+1 dimensional viscous hydrodynamics and discuss how the constraint extends to higher dimensions.

2. 0+1 Dimensional Conformal 2nd-Order Viscous Hydrodynamics

We consider a conformal fluid that is expanding in a boost invariant manner along the longitudinal direction with a uniform energy density along the transverse plane. For this 0+1 dimensional model, the 2nd-order viscous hydrodynamic equations are given by (Muronga 2004, Baier et. al. 2007):

$$\partial_\tau \epsilon = -\frac{\epsilon + p}{\tau} + \frac{\Pi}{\tau},$$
$$\partial_\tau \Pi = -\frac{\Pi}{\tau_\pi} + \frac{4\eta}{3\,\tau_\pi \tau} - \frac{4}{3\,\tau}\Pi - \frac{\lambda_1}{2\,\tau_\pi\,\eta^2}(\Pi)^2 .$$

The coefficients η, τ_π and λ_1 are the transport coefficients. Recent studies seem to show that in general, these coefficients don't follow universal relations and their values haven't been completely determined at higher orders.
Note that in either the strong or weak coupling limit the coefficients τ_π and λ_1 are proportional to $\tau_\pi \propto T^{-1}$ and $\lambda_1 \propto \bar{\eta}^2 s/T$. This suggests that we can parametrize both coefficients as:

$$\tau_\pi = \frac{c_\pi}{T} \qquad \lambda_1 = c_{\lambda_1}\bar{\eta}^2\left(\frac{s}{T}\right).$$

where $\bar{\eta} \equiv \eta/s$. The dimensionless numbers $\bar{\eta}$, c_π and c_{λ_1} carry all of the information about the particular coupling limit we are considering. In our analysis, we use the values of these parameters as follows:

Parameter	Weakly-coupled QCD	Strongly-coupled $\mathcal{N}=4$ SYM
$\bar{\eta} \equiv \eta/s$	$10/(4\pi)$	$1/(4\pi)$
c_π	$6\bar{\eta}$	$(2-\log 2)/(2\pi)$
c_1	$9/2$	2

Specification of equation of state and dimensionless variables

We will assume an ideal equation of state and conformal limit, i.e., $\epsilon = 3p$. For QCD with three colors and two flavors, the thermodynamic variables are related as:

$$p = \frac{37\,\pi^2}{90}T^4 \Rightarrow \epsilon = (T/\gamma)^4, \quad \text{with} \quad \gamma \equiv \left(\frac{30}{37\,\pi^2}\right)^{1/4},$$
$$s = \frac{4}{3\gamma}\epsilon^{3/4}.$$

Using the ideal gas equation of state, the parametrization of τ_π and λ_1 can be rewritten in terms of the energy density ϵ:

$$\tau_\pi = \frac{c_\pi}{\gamma\,\epsilon^{1/4}} \qquad \lambda_1 = \frac{4}{3\gamma^2}c_{\lambda_1}\bar{\eta}^2\,\epsilon^{1/2}.$$

To remove the dimensionful scales, we define the dimensionless variables $\bar{\epsilon} \equiv \epsilon/\epsilon_0$, $\overline{\Pi} \equiv \Pi/\epsilon_0$ and $\bar{\tau} \equiv \tau/\tau_0$. After replacing these dimensionless variables, we rewrite the fluid equations:

$$\bar{\tau}\,\partial_{\bar{\tau}}\bar{\epsilon} + \frac{4}{3}\bar{\epsilon} - \overline{\Pi} = 0,$$
$$\overline{\Pi} + \frac{c_\pi}{\gamma\,k\,\bar{\epsilon}^{1/4}}\left[\partial_{\bar{\tau}}\overline{\Pi} + \frac{4\overline{\Pi}}{3\,\bar{\tau}}\right] - \frac{16\,\bar{\eta}\,\bar{\epsilon}^{3/4}}{9\,\gamma\,k\,\,\bar{\tau}} + \frac{3\,c_{\lambda_1}}{8}\frac{\overline{\Pi}^2}{\bar{\epsilon}} = 0.$$

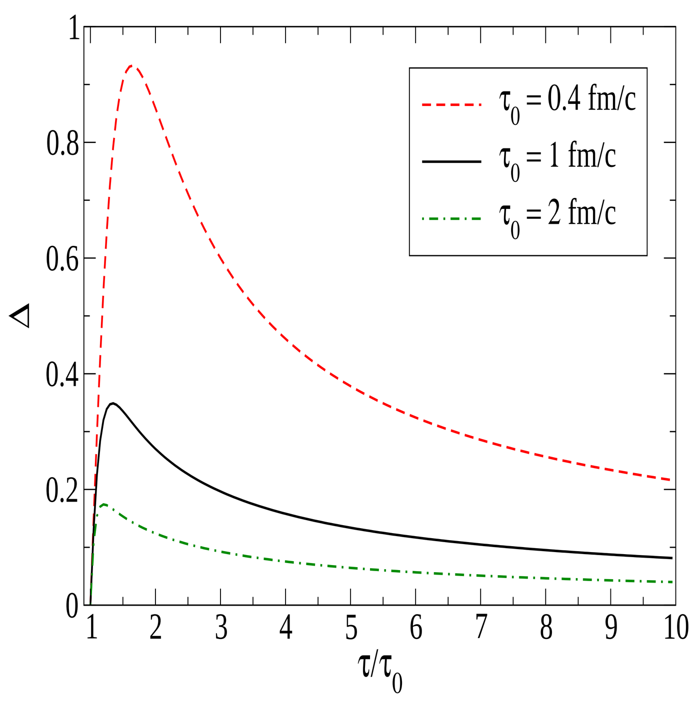
where $k \equiv \tau_0\epsilon_0^{1/4}$. The boundary conditions are specified at $\bar{\tau} = 1$ where $\bar{\epsilon} = 1$ and $\overline{\Pi}(\bar{\tau}=1) = \overline{\Pi}_0$ which is a free parameter. All information about the initial proper-time and energy density is encoded in the parameter k and all information about the equation of state is encoded in the parameter γ.

Momentum-space anisotropy

The degree of momentum-space isotropy of the fluid can be measured through the dimensionless parameter Δ:

$$\Delta \equiv \frac{P_T}{P_L} - 1,$$

In the 0+1 dimensional model of viscous hydro, the effective transverse pressure is $P_T = p + \Pi/2$ and the effective longitudinal pressure as $P_L = p - \Pi$. In the case of an ideal equation of state, Δ can be expressed in terms of our dimensionless variables:



$$\Delta = \frac{9}{2}\left(\frac{\overline{\Pi}}{\bar{\epsilon} - 3\overline{\Pi}}\right) \Rightarrow \Delta(\bar{\tau}=1) \equiv \Delta_0 = \frac{9}{2}\left(\frac{\overline{\Pi}_0}{1 - 3\overline{\Pi}_0}\right)$$

Positivity of the longitudinal pressure requires $\Delta \neq \infty$ at any time during the evolution of the plasma. However, in order to apply viscous hydro it is required that $|\overline{\Pi}| \ll \bar{p}$. This can be accomplished by requiring that $-\alpha\,\bar{p} < \overline{\Pi} < \alpha\,\bar{p}$, where $0 \le \alpha \le 1$. For general α requires $\Delta_- \le \Delta \le \Delta_+$ where

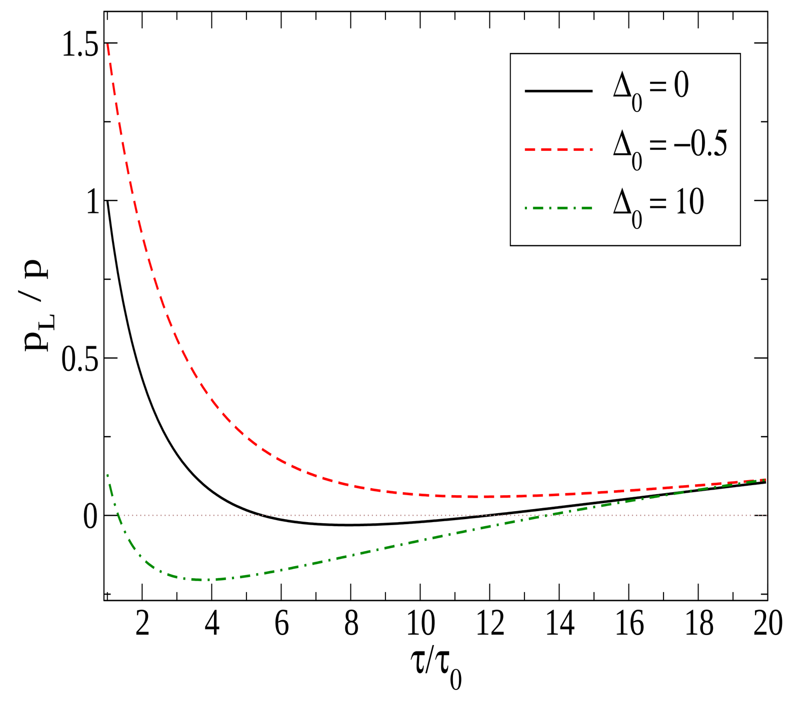
$$\Delta_\pm \equiv \pm\frac{3}{2}\left(\frac{\alpha}{1 \mp \alpha}\right).$$

For example, requiring $\alpha = 1/3$ we would find the constraint $-3/8 \le \Delta_\alpha \le 3/4$.

3. Results

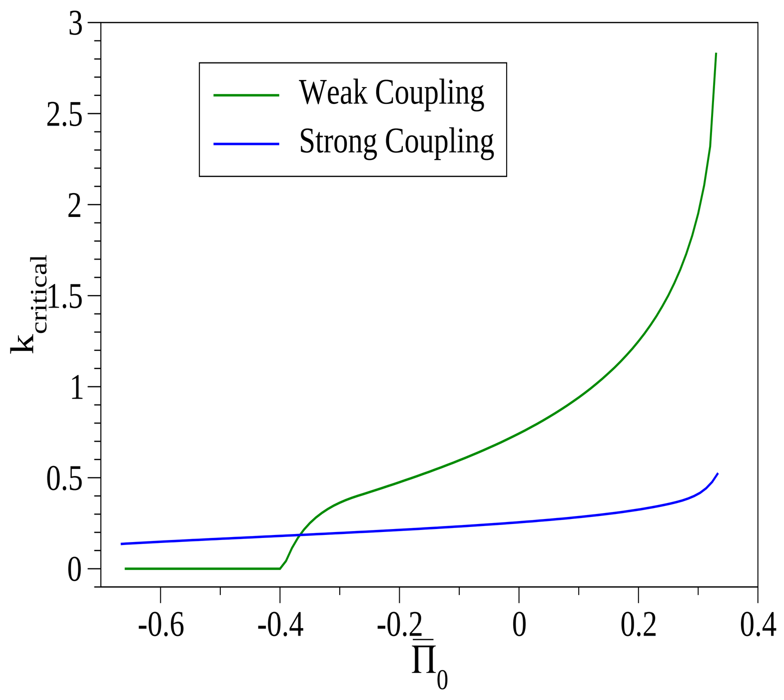
Negativity of the Longitudinal Pressure

The negativity of the longitudinal pressure indicates that the expansion is breaking down. Implicitly, it is assumed that $\Pi \ll \bar{p}$. The point at which $P_L = 0$ is the point at which, Π, is equal in magnitude to the background around which one is expanding. This means that the perturbation is no longer a small correction and higher order corrections could become important. Therefore negative longitudinal pressure signals regions of parameter space where one cannot trust 2nd-order viscous hydrodynamical solutions.
From a transport theory point of view it indicates that something is unphysical about the simulation since in transport theory all components of the pressure are positive definite.



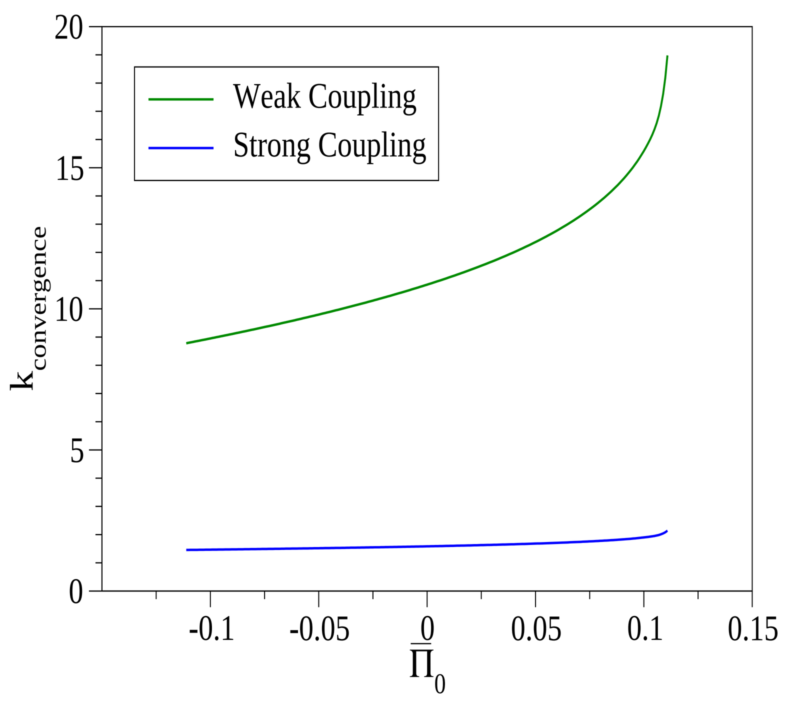
Determining the critical line

Once one chooses the transport coefficients and eqn. of state, the only remaining freedom comes from the dimensionless coefficient $k = \tau_0\epsilon_0^{1/4}$ and the initial shear $\overline{\Pi}_0$ for the 0+1 dimensional case. For a given $\overline{\Pi}_0$, we find that for k below a certain value, the system exhibits a negative longitudinal pressure. Above the critical value of k the longitudinal pressure is positive definite at all times. Since k is proportional to the assumed initial simulation time τ_0 increasing k with fixed initial energy density corresponds to increasing τ_0. Assuming fixed initial temperature, for an initially prolate distribution, one can start the simulation at earlier times. In the 0+1 dimensional case, $\tau_0 > \gamma\,k_{\rm critical}\,T_0^{-1}$ and our result can be used to set a bound on this product.



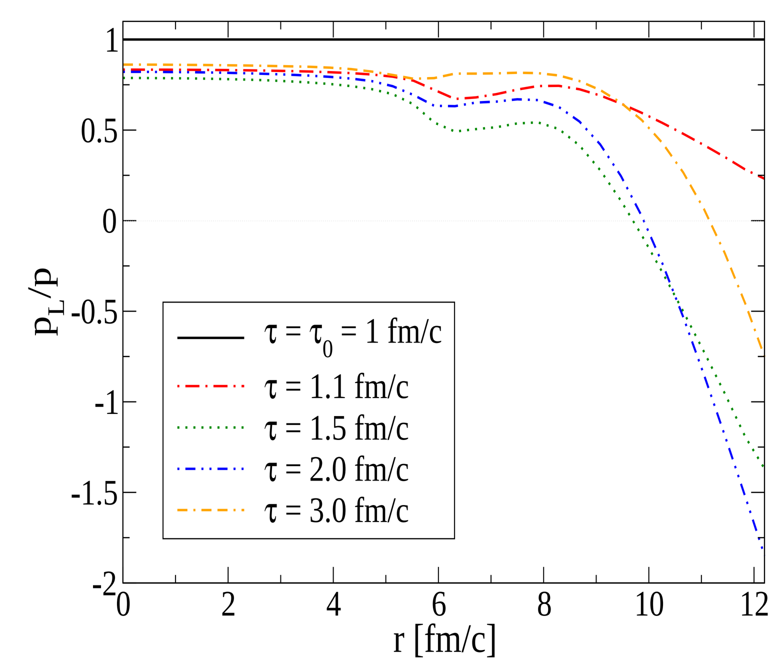
Determining the convergence line

The requirement that the $P_L(\tau) > 0$ gives a weak constraint because of it merely requires that $\overline{\Pi} < \bar{p}$. A stronger constraint is obtained by requiring instead $-\alpha\,\bar{p} \le \overline{\Pi} \le \alpha\,\bar{p}$ and then using this to constrain the possible initial time and energy density. For $\alpha = 1/3$, the initial values of $\overline{\Pi}_0$ are constrained to be between $-1/9 \le \overline{\Pi}_0 \le 1/9$. For a given $\overline{\Pi}_0$ in this range we find that for k below a certain value we cannot satisfy the stronger constraint. This point in k defines the "convergence" value of k or $k_{\rm convergence}$. Above $k = k_{\rm convergence}$ the shear satisfies the constraint $-\bar{p}/3 \le \overline{\Pi} \le \bar{p}/3$. In the case of an ideal QCD equation of state and assuming $\overline{\Pi}_0 = 0$, the constraint is that $\tau_0 > \gamma\,k_{\rm convergence}\,T_0^{-1}$. For the strong coupling case, $\tau_0 > 0.85\,T_0^{-1}$. For a weakly coupled case $\tau_0 > 5.9\,T_0^{-1}$.



Extension to higher dimensions

As one goes away from the center of the hot and dense matter, the energy density drops and, one would find that at a finite distance from the center the condition $k > k_{\rm critical}$ would be violated by the initial conditions. In these regions of space, hydrodynamics would then predict an infinitely large anisotropy parameter, Δ, casting doubt on the reliability of the hydrodynamic assumptions. For a 2+1 dimensional viscous hydrodynamics we found that for RHIC energies in the strong coupling limit, P_L becomes negative for transverse radius $r \geqslant 6.8 fm$ at very early times. At radii larger than this value it is possible that higher order corrections are large and therefore the applicability of 2nd-order viscous hydrodynamics becomes questionable.



4. Conclusions

- We have derived two general criteria that can be used to assess the applicability of 2nd-order conformal viscous hydrodynamics to relativistic heavy-ion collisions. We did this by simplifying to a 0+1 dimensional system undergoing boost invariant expansion and then (a) requiring the longitudinal pressure to be positive during the simulated time or (b) requiring a convergence criterium that $|\Pi| < p/3$ during the simulated time. These requirements lead to a non-trivial relation between the possible initial simulation time τ_0, the initial energy density ϵ_0, and the initial value of the fluid shear tensor, Π_0.
- The constraints derived here were then shown to provide guidance for where one might expect 2nd-order viscous hydrodynamics to be a good approximation in higher-dimensional cases.