Calculating transport coefficient in a parton cascade

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Solving the Boltzmann equation for on-shell gluons (and quarks) using Monte Carlo techniques
> 2<->3 processes included (detailed balance)
stochastic interpretation of collision rates

----- in particular for this work -----------------------------
>> Boltzmann gas of gluons(only)
>> 1-Dim Bjorken expanding geometry
>> pQCD calculated crosssections
Thermalization of a CGC in BAMPS

Initial condition: simple form of CGC

\[ f(x,p) = \frac{c}{\alpha_s N_c t_{ini}} \frac{1}{\delta(p_z) \Theta(Q_s^2 - p_t^2)} \]

\[ \frac{dN}{d\eta} = c \pi R^2 \frac{N_c^2 - 1}{4 \pi^2 \alpha_s N_c} Q_s^2 \]

\( Q_s = 2, 3, 4 \text{ GeV}, \ \alpha_s = 0.1, 0.2, 0.3 \)

\( \alpha_s = 0.3 \)

- \( t_{th}(Q_s = 2 \text{ GeV}) = 1.2 \text{ fm/c} \)
- \( t_{th}(Q_s = 3 \text{ GeV}) = 0.75 \text{ fm/c} \)
- \( t_{th}(Q_s = 4 \text{ GeV}) = 0.55 \text{ fm/c} \)

\( Q_s = 3 \text{ GeV} \)

- \( t_{th}(\alpha_s = 0.1) = 1.75 \text{ fm/c} \)
- \( t_{th}(\alpha_s = 0.2) = 1.0 \text{ fm/c} \)
- \( t_{th}(\alpha_s = 0.3) = 0.75 \text{ fm/c} \)
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\( Q_s = 2, \ 3, \ 4 \ \text{GeV}, \ \alpha_s = 0.1, \ 0.2, \ 0.3 \)

\( \text{RHIC} \quad \text{LHC} \)

Comparison with Bottom-Up:

\[ t_{th} \sim \alpha_s^{-\frac{13}{5}} \text{ wrong} \quad t_{th} \sim \frac{1}{Q_s} \text{ fulfilled} \]

BAMPS:

\[ t_{th} \sim \alpha_s^{-2} (\ln \alpha_s)^{-2} Q_s^{-1} \]

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\( \alpha_s = 0.3 \)

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Thermalization of a CGC in BAMPS: (quasi) ideal hydrodynamic behavior

Do we observe an ideal hydro behavior and how large is the viscosity?
Can we compare to hydrodynamic models?
Grad’s method:

> Derivation of causal hydrodynamic equations from the Boltzmann equation.

> How can we calculate the transport coefficients using this formalism?

Assume a small deviation of distribution function $f(x,p)$ from equilibrium

Develop the distribution function up to second order in momentum

Consider second law of thermodynamics $\partial_\mu S^\mu \geq 0$ to identify the transport coefficients

 Need to calculate momenta of the Boltzmann equation
References:


Zhe Xu, Carsten Greiner, *Phys. Rev. C.* 76, 024911
From Boltzmann Equation to relativistic dissipative hydro.

\[ p^\mu \partial_\mu f(x,p) = C[f(x,p)] \]

**Boltzmann Equation**

**Entropy four-current**

\[ S^\mu(x) = - \int dw \ p^\mu f(x,p) \ln \left( \frac{g}{(2\pi)^3} f(x,p) \right) \]

Definition of entropy four-current in kinetic theory

\[ \partial_\mu S^\mu(x) = - \frac{g}{(2\pi)^3} \int dw \ p^\mu \partial_\mu f(x,p) \ln [f(x,p)] = \]

\[ = - \frac{g}{(2\pi)^3} \int dw \ p^\mu \partial_\mu f(x,p) \ln [f(x,p)] - \frac{g}{(2\pi)^3} \int dw \ p^\mu f(x,p) \frac{\partial_\mu f(x,p)}{f(x,p)} \]

\[ \int dw \ p^\mu \partial_\mu f(x,p) = \int dw \ C[f(x,p)] = 0 \]

(intrinsic symmetry of the collision term)

\[ = - \frac{g}{(2\pi)^3} \int dw \ p^\mu \partial_\mu f(x,p) \ln [f(x,p)] \]
From Boltzmann Equation to relativistic dissipative hydro.

\[
\ln[f(x,p)] = y(x,p) = \phi(x,p) + y^{eq}(x,p)
\]

assuming \( \Phi \) is small

\[
f(x,p) = e^y = e^{y^{eq}}.e^\phi = f^{eq}(x,p)(1 + \phi(x,p))
\]

up to 2\textsuperscript{nd} order in momentum

\[
\phi(x,p) = \epsilon(x) - \epsilon_\mu(x)p^\mu + \epsilon_\mu\nu(x)p^\mu p^\nu
\]

\[
\partial_\mu S^\mu(x) = -\frac{g}{(2\pi)^3} \int dwp^\mu \partial_\mu f(x,p) \ln[f(x,p)] = -\frac{g}{(2\pi)^3} \int dwC[f]y(x,p) =
\]

\[
= -\frac{g}{(2\pi)^3} \int dwC[f].(\epsilon(x) - \epsilon_\mu(x)p^\mu + \epsilon_\mu\nu(x)p^\mu p^\nu + y^{eq}(x,p))
\]

vanishing momenta of collision term because of

\[
\partial_\mu N^\mu = 0 \land \partial_\mu T^{\mu\nu} = 0 \land \text{energy conservation}
\]
From Boltzmann Equation to relativistic dissipative hydro.

\[ \partial_\mu S^\mu(x) = -\frac{g}{(2\pi)^3} \int dw \ p^\mu \partial_\mu f(x, p) \ln[f(x, p)] = \]

\[ = -\frac{g}{(2\pi)^3} \int dw C[f] \cdot (\epsilon(x) - \epsilon_\mu(x) p^\mu + \epsilon_{\mu\nu}(x) p^\mu p^\nu + y^{eq}(x, p)) \]

\[ \partial_\mu S^\mu = -\frac{g}{(2\pi)^3} \epsilon_{\mu\nu}(x) \int dw \ p^\mu p^\nu C[f] = -\epsilon_{\mu\nu} P^{\mu\nu} = \beta (\zeta^{-1} \Pi^2 - \lambda^{-1} q^\alpha q_\alpha + (2\eta)^{-1} \pi^{\alpha\beta} \pi_{\alpha\beta}) \]

can be calculated from cascade

What is \( \epsilon_{\mu\nu} \)?
We can decompose $N^\mu, T^{\mu\nu}$ and higher momenta in terms of $J_{mn}$, $u^\mu$ and $\varepsilon, \varepsilon^\mu, \varepsilon_{\mu\nu}$.

Now: take projections of $N^\mu, T^{\mu\nu}$... and use the definitions of the dissipative fluxes.

$$\epsilon_{\mu\nu} = A_2(3u_\mu u_\nu - \Delta_{\mu\nu})\Pi - B_1 u_{(\nu}q_{\mu)} + C_0 \pi_{\mu\nu}$$

$$\epsilon_{\mu} = A_1 u_{\mu} \Pi - B_0 q_\mu$$

$$\epsilon = A_0 \Pi$$

The entropy production was

$$\partial_\mu S^\mu = -\frac{g}{(2\pi)^3} \epsilon_{\mu\nu}(x) \int dw p^\mu p^\nu C[f] = -\epsilon_{\mu\nu} P^{\mu\nu} = \beta \varepsilon_{\mu\nu} = \beta (\zeta^{-1} \Pi^2 - \lambda^{-1} q^\alpha q_\alpha + (2\eta)^{-1} \pi^{\alpha\beta} \pi_{\alpha\beta})$$

To identify the transport coefficients, we need to decompose $P^{\mu\nu}$:

$$P^{\mu\nu} = \frac{4}{3} C_\Pi A_2 (3u^\mu u^\nu - \Delta^{\mu\nu})\Pi + 2C_q B_1 q^{(\mu}u_{\nu)} + \frac{1}{5} C_\pi C_0 \pi^{<\mu\nu>}$$

$C_\Pi, C_q, C_\pi$ are unknown coefficients, involving integrals of the collision term.
Momenta of the distribution function: calculating \(J_{mn}\)

\[N_{eq} = \int f^{eq} p^\mu dw = J_{10} u^\mu\]

\[T_{eq}^{\mu\nu} = \int f^{eq} p^\mu p^\nu dw = J_{20} u^\mu u^\nu - J_{21} \Delta^{\mu\nu}\]

\[F_{eq}^{\lambda\mu\nu} = \int f^{eq} p^\lambda p^\mu p^\nu dw = J_{30} u^\lambda u^\mu u^\nu - 3 J_{31} \Delta^{(\lambda\mu\nu)}\]

using \(u^\mu u_\mu = 1\) one obtains from the first eq: \(u^\mu N^{\mu}_{eq} = \int f^{eq} u^\mu p^\mu dw = J_{10} = \frac{16}{\pi^2} T^3\)

Lorentz scalar. Evaluating in the LRF, \(u = (1,0,0,0)\)

\[j_{20} = \frac{48}{\pi^2} T^4;\]

\[j_{30} = \frac{192}{\pi^2} T^5;\]

\[j_{40} = \frac{960}{\pi^2} T^6;\]

\[j_{21} = \frac{16}{\pi^2} T^4;\]

\[j_{31} = \frac{64}{\pi^2} T^5;\]

\[j_{41} = \frac{320}{\pi^2} T^6;\]

for massless bosons with 0 chemical potential:

\[J_{nk} = \frac{16}{2 \pi^2 (2k+1)!!} T^{n+1} \Gamma(n+2) \zeta(n+1)\]
Transport coefficients

\[ \partial_\mu S^\mu = -\epsilon_{\mu\nu} P^{\mu\nu} = \beta (\zeta^{-1} \Pi^2 - \lambda^{-1} q^\alpha q_\alpha + (2\eta)^{-1} \pi^{\alpha\beta} \pi_{\alpha\beta}) \]

\[ \epsilon_{\mu\nu} = A_2 (3u_\mu u_\nu - \Delta_{\mu\nu}) \Pi - B_1 u_{(\mu} q_{\nu)} + C_0 \pi_{\mu\nu} \]

\[ P^{\mu\nu} = \frac{4}{3} C_\Pi A_2 (3u^\mu u^\nu - \Delta^{\mu\nu}) \Pi + 2C_q B_1 q^{(\mu} u_{\nu)} + \frac{1}{5} C_\pi C_0 \pi^{<\mu\nu>} \]

\[ \eta = \frac{-5\beta}{2C_0^2 C_\pi} \]

Taking projection of \( P^{\mu\nu} \):

\[ P^{<\mu\nu>} = \frac{1}{5} C_\pi C_0 \pi^{<\mu\nu>} = \frac{1}{5} C_\pi C_0 T^{<\mu\nu>} \]

\[ C_\pi = \frac{5P^{<\mu\nu>}}{C_0 T^{<\mu\nu>}} \]

\[ \pi^{<\mu\nu>} = \left[ \frac{1}{2} (\Delta_\alpha^\mu \Delta_\beta^\nu - \Delta_\beta^\mu \Delta_\alpha^\nu) - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta} \right] \pi^{\alpha\beta} = \left[ \frac{1}{2} (\Delta_\alpha^\mu \Delta_\beta^\nu - \Delta_\beta^\mu \Delta_\alpha^\nu) - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta} \right] T^{\alpha\beta} \]
Transport coefficients

\[ \eta = \frac{-\beta T \langle \mu \nu \rangle}{2C_0 P \langle \mu \nu \rangle} \quad \text{with} \quad C_0 = \frac{1}{2J_{42}} \quad \beta = \frac{1}{T} \left( \frac{\pi^2 e}{48} \right)^{-\frac{1}{4}} \]

\[ \zeta = \frac{\beta}{16A_2 C_\Pi} \quad \text{with} \quad u_\nu u_\lambda P^{\nu\lambda} = 4C_\Pi A_2 \Pi \]

\[ \lambda = \frac{\beta}{B_1 C_q} \quad \text{with} \quad \Delta^{\mu}_\nu u_\lambda P^{\nu\lambda} = C_q B_1 q^\mu \]

vanish for a massless gas

Relaxation times

\[ \tau_\pi = 2 \eta \beta_2 \]

\[ \tau_\Pi = \zeta \beta_0 \]

\[ \tau_q = \kappa T \beta_1 \]

with \[ \beta_2 = \frac{1}{2} \frac{J_{52}}{J_{42}^2} = \frac{3}{64} \frac{\pi^2}{T^4} \]

Can be derived from \[ \partial_\mu F^{\mu\nu\lambda} = P^{\nu\lambda} \]
Implementation in the BAMPS

In the central rapidity bin:

\[
\eta = \frac{-\beta T}{2C_0} \frac{<\mu v>}{P_{<\mu v>}}
\]

\[
u_{\mu} = (\gamma, 0, 0, \frac{z}{t}) = (\cosh \eta, 0, 0, \tanh \eta) \approx (1, 0, 0, 0)
\]

\[
\eta = \beta \frac{2T^{33} - T^{22} - T^{11}}{2C_0 (P^{11} + P^{22} - 2P^{33})}
\]

with

\[
C_0 = \frac{1}{2J_{42}} \quad \beta = \frac{1}{T} = \left(\frac{\pi^2 e}{48}\right)^{-\frac{1}{4}}
\]

\[
J_{42} = \frac{64}{\pi^2} T^6
\]
Calculating shear viscosity in BAMPS:

\[ \eta = \beta \frac{2T^{33} - T^{22} - T^{11}}{2C_0 (P^{11} + P^{22} - 2P^{33})} \]

\[ P^{\mu\nu} = \int \frac{d^3p}{(2\pi)^3} p^{\mu} p^{\nu} C[f] \]

\[ P^{(22)}_{\mu\nu} = \int \frac{d^3p_1}{(2\pi)^3} p_1^{\mu} p_1^{\nu} \frac{1}{2E_1} \int d\Gamma_2 \frac{1}{2} \int d\Gamma' f_1 f_2 |M_{1'2'\rightarrow12}|^2 (2\pi^4) \delta^{(4)}(p_1 + p_2 - p_1 - p_2) \]

\[ -\int \frac{d^3p_1}{(2\pi)^3} p_1^{\mu} p_1^{\nu} \frac{1}{2E_1} \int d\Gamma_2 f_1 f_2 \frac{1}{2} \int d\Gamma' f_2' |M_{1'2'\rightarrow12}|^2 (2\pi^4) \delta^{(4)}(p_1 + p_2' - p_1 - p_2') \]

\[ \text{with} \quad d\Gamma = \frac{d^3p}{(2\pi)^3 2E} \]

similar (but more complicated) for \( P^{(23)}_{\mu\nu}, P^{(32)}_{\mu\nu} \)
Calculating shear viscosity in BAMPS:

Navier-Stokes with Bjorken scaling

\[ \eta = \frac{T}{4} \left( T_{11} + T_{22} - 2T_{33} \right) \]

Grad's method

\[ \eta = \beta \frac{2T^{33} - T^{22} - T^{11}}{2C_0(P^{11} + P^{22} - 2P^{33})} \]

with

\[ C_0 = \frac{1}{2J_{42}} \quad \beta = \frac{1}{T} = \left( \frac{\pi^2 e}{48} \right)^{-\frac{1}{4}} \]

\[ J_{42} = \frac{64}{\pi^2} T^6 \]

\[ \alpha_s = 0.3 \]

\[ t_{th}(Q_s=2 \text{ GeV}) = 1.2 \text{ fm} \]

\[ t_{th}(Q_s=3 \text{ GeV}) = 0.75 \text{ fm} \]
Shear to entropy density & relaxation times:

\[
\alpha_s = 0.3
\]

\[
\begin{align*}
\eta/s &= \begin{cases} 
0.3 & Q_s = 2 \text{ GeV} \\
0.25 & Q_s = 3 \text{ GeV} \\
0.15 & Q_s = 4 \text{ GeV}
\end{cases} \\
\tau_\pi &= \begin{cases} 
1.2 & Q_s = 2 \text{ GeV} \\
0.75 & Q_s = 3 \text{ GeV} \\
0.55 & Q_s = 4 \text{ GeV}
\end{cases}
\end{align*}
\]
Taking a more realistic initial condition
(modified form of the Kharzeev-Levin-Nardi approach, H.J. Drescher, Y.Nara,

KLN: initial condition chosen to match the charged hadron multiplicity @RHIC (PHOBOS data)
high $p_T$ "tails" ($p_T > Q_S$) are included in KLN approach.
Summary:

> 2\textsuperscript{nd} order hydrodynamic eqs can be derived from Boltzmann equation (grad's method)

> Transport coefficients (shear viscosity, shear relaxation time) can calculated in a partonic cascade with CGC initial conditions

> To do: 

\textbf{Comparison to hydro model.}
Momenta of the distribution function

\[ N_{eq} = \int f^{eq} p^\mu dw = J_{10} u^\mu \]

\[ T_{eq}^{\mu\nu} = \int f^{eq} p^\mu p^\nu dw = J_{20} u^\mu u^\nu - J_{21} \Delta^{\mu\nu} \]

\[ F_{eq}^{\lambda\mu\nu} = \int f^{eq} p^\lambda p^\mu p^\nu dw = J_{30} u^\lambda u^\mu u^\nu - 3 J_{31} \Delta^{(\lambda \mu \nu)} \]

can be done analytically

\[
\begin{align*}
N^\mu &= \int f \, dw = N_{eq}^\mu + \epsilon \int f^{eq} p^\mu dw - \epsilon \nu \int f^{eq} p^\mu p^\nu dw + \epsilon_{\nu \lambda} \int f^{eq} p^\nu p^\lambda p^\mu dw \\
T^{\mu\nu} &= \int f \, dw = T_{eq}^{\mu\nu} + \epsilon \int f^{eq} p^\mu p^\nu dw - \epsilon \lambda \int f^{eq} p^\lambda p^\mu p^\nu dw + \epsilon_{\lambda \rho} \int f^{eq} p^\lambda p^\rho p^\mu p^\nu dw \\
&\vdots
\end{align*}
\]

\[ \Rightarrow N^\mu, \ T^{\mu\nu} \text{ in terms of } u^\mu, \ J_{mn}; \ \epsilon, \epsilon_{\nu \lambda}, \epsilon_{\mu \nu} \text{ only!} \]
Definitions of dissipative fluxes

\( V^\mu = \Delta^\mu_\nu \delta N^\nu \) \hspace{1cm} \text{particle drift flux}

\( W^\mu = \Delta^\mu_\nu u_\rho \delta T^{\nu\rho} \) \hspace{1cm} \text{energy flux}

\( q^\mu = W^\mu - h V^\mu \) \hspace{1cm} \text{heat flux}

\( \Pi = -\frac{1}{3} \Delta^\nu_\mu \delta T^{\mu\nu} \) \hspace{1cm} \text{bulk pressure}

\( \pi^{\nu\mu} = \delta T^{<\mu
\nu>} \) \hspace{1cm} \text{shear stress tensor}

\[ \Delta^{\nu\mu} = g^{\nu\mu} - u^{\nu} u^{\mu}, \quad g^{\nu\mu} = \text{diag}(1, -1, -1, -1) \]

\[ h = \frac{e + p}{e} \]

Projector \( <..> : \quad A^{<\mu
\nu>} = \left[ \frac{1}{2} (\Delta^{\mu}_\alpha \Delta^{\nu}_\beta + \Delta^{\mu}_\beta \Delta^{\nu}_\alpha) - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\alpha\beta} \right] A^{\alpha\beta} \]