Multiplicity Fluctuations in Relativistic Gases

Michael Hauer

Helmholtz Research School, University of Frankfurt, Frankfurt, Germany

IFJ Seminar, Kraków, 16 Nov 07
Enhanced fluctuations are expected near a critical point of strongly interacting matter, or as a result of the onset of deconfinement.

Correlations, i.e. susceptibilities, are different in HRG and QGP.

Good experimental data are becoming available.

Calculate baseline (or statistical) fluctuations.

Test the statistical hadronization model.
Introduction

Statistical Ensembles

\[ Z^{GCE}(V, T, \mu) \]

Grand Potential

\[ \Omega = -T \ln Z^{GCE} \]

Helmholtz Free Energy

\[ F = -T \ln Z^{CE} \]

Entropy

\[ S = \ln Z^{MCE} \]

\[ \mu^{CE} = \left( \frac{\partial F}{\partial Q} \right)_{V, T} \]

\[ \frac{1}{T_{MCE}} = \left( \frac{\partial S}{\partial E} \right)_{V, Q} \]

Thermodynamic Limit: \( V, E, Q \to \infty \)

\[ \frac{E}{V} = \text{const}, \quad \frac{Q}{V} = \text{const}, \quad \Rightarrow \quad T_{MCE} \to T, \quad \mu^{CE} \to \mu \]
GCE Partition Function

\[
Z^{GCE}(V, T) = \exp \left[ Vg \int \frac{d^3p}{(2\pi)^3} e^{-\frac{|p|}{T}} \right] = \exp \left[ Vg \frac{T^3}{\pi^2} \right]
\]

All Micro-states with N Particles

\[
Z^N(V, T) = \int_{-\pi}^{\pi} \frac{d\phi_N}{2\pi} e^{-iN\phi_N} \exp \left[ Vg \frac{T^3}{\pi^2} e^{i\phi_N} \right] = \left( Vg \frac{T^3}{\pi^2} \right)^N
\]

GCE Multiplicity Distribution is Poissonian

\[
P_{GCE}(N) = \frac{Z^N(V, T)}{Z^{GCE}(V, T)} = \frac{\left( Vg \frac{T^3}{\pi^2} \right)^N}{N!} \exp \left( -Vg \frac{T^3}{\pi^2} \right)
\]
Ultrarelativistic Gas of Neutral Particles

All Micro-states with \( N \) Particles and Energy \( E \)

\[
Z^{N,E}(V, T) = \int_{-\pi}^{\pi} \frac{d\phi_N}{2\pi} \int_{-\infty}^{\infty} \frac{d\phi_E}{2\pi} e^{-iN\phi_N} e^{-iE\phi_E} \exp \left[ Vg \int \frac{d^3p}{(2\pi)^3} e^{-|p|/T} e^{ip|\phi_E e^{i\phi_N}} \right] \\
= \left( \frac{gV}{\pi^2} \right)^N \frac{E^{3N-1}}{N!(3N-1)!} e^{-E/T} = Z^{MCE}(V, E, N) e^{-E/T}
\]

All Micro-states with Energy \( E \)

\[
Z^E(V, T) = \sum_{N=1}^{\infty} Z^{E,N}(V, T) \quad \text{or} \quad Z^{MCE}(V, E) = \sum_{N=1}^{\infty} Z^{MCE}(V, E, N)
\]

MCE Multiplicity Distribution

\[
P_{MCE}(N) = \frac{Z^{N,E} e^{+E/T}}{Z^E e^{+E/T}} = \frac{Z^{MCE}(V,E,N)}{Z^{MCE}(V,E)}
\]

MCE \( P(N) \) can be expressed through conditional GCE \( P(N|E)! \)

M.H., V.V. Begun, M.I. Gorenstein, arXiv:0706.3290 [nucl-th]
**What is the role of T in MCE?**

**Partition Function**

\[
Z^E(V, T) \equiv Z^{MCE}(V, E) e^{-\frac{E}{T}}
\]

**Entropy**

\[
S = \ln \left( Z^E e^{\frac{E}{T}} \right)
\]

**Determine Equilibrium Temperature**

\[
\left( \frac{\partial S}{\partial E} \right)_V = \frac{\partial Z^E}{\partial E} e^{\frac{E}{T}} + \frac{1}{T} Z^E e^{\frac{E}{T}} = \frac{1}{T_{MCE}}
\]

\[
T = T_{MCE} \text{ implies: } \frac{\partial Z^E}{\partial E} = 0
\]

\[
\Rightarrow \text{Maximize GCE Partition Function for Energy E}
\]

**GCE Partition Function**

\[
Z^{GCE}(V, T) = 1 + \int_0^{\infty} dE \ Z^{MCE}(V, E) e^{-\frac{E}{T}} = 1 + \int_0^{\infty} dE \ Z^E(V, T)
\]
What is the advantage of introducing $T$ in MCE?

OR:

Why is it of advantage to define MCE multiplicity distributions through joint GCE distributions?

Principle Problem:

In CE and MCE calculations one has to deal with a heavily oscillating (or even irregular) integrand.

- Our version is however very smooth!
- Main contribution comes from small region around the origin.
- Analytical expansion is possible
- Numerical integration for large systems becomes feasible!
General Relativistic Ideal Multi-specie Hadron Gas

Fourier Spectral Analysis of GCE Partition Function

\[ Z^{Q^j, E^k}(V, T, \mu_j) = \left[ \prod_{j=1}^{3} \int_{-\pi}^{\pi} \frac{d\phi_j}{2\pi} e^{-iQ^j \phi_j} \right] \left[ \prod_{k=1}^{4} \int_{-\infty}^{\infty} \frac{d\phi_k}{2\pi} e^{-iE^k \phi_k} \right] \exp \left[ V \sum_l \psi_l(\phi_j, \phi_k) \right] \]

Single Particle Partition Function

\[ \psi_l(\phi_j, \phi_k) = \frac{g_l}{(2\pi)^3} \int d^3p \ln \left( 1 \pm e^{-\sqrt{m^2 + p^2 - \mu_l}} e^{i\xi_j \phi_j} e^{i\xi_k \phi_k} \right)^\pm 1 \]

It can be shown that generally:

\[ Z^{Q^j, E^k}(V, T, \mu_j) = Z^{MCE}(V, Q^j, E^k) e^{\frac{Q^j \mu_j}{T}} e^{-\frac{E^k}{T}} \]

In the large volume limit \( Z^{Q^j, E^k}(V, T, \mu_j) \) converges to a Multivariate-Normal-Distribution. Finite Volume corrections are given in the form of Hermite polynomials of low order.

Definitions:

\[ Q^j = (B, S, Q) \]
\[ E^k = (E, P_x, P_y, P_z) \]
\[ q^j_l = (b_l, s_l, q_l) \]
\[ \varepsilon^k_l = (\varepsilon_l, p_l,x, p_l,y, p_l,z) \]

M.H., V.V.Begun, M.I. Gorenstein, arXiv:0706.3290 [nucl-th]
Gram-Charlier Expansion

**General Hadron Gas Partition Function**

\[
Z^Q_j \simeq \frac{Z^{GCE}}{V^{J/2} \det |\sigma|} \left[ \prod_{j=1}^J \int_{-\infty}^{\infty} \frac{d\theta_j}{2\pi} \right] \exp \left[ -i\xi^j \theta_j - \frac{\theta_j^2}{2V} + \sum_{n=3}^{\infty} \frac{\lambda_n^{j_1,j_2,\ldots,j_n}}{n!} \theta_{j_1} \theta_{j_2} \cdots \theta_{j_n} \right]
\]

**Normalized Cumulants**

\[
\lambda_n^{j_1,j_2,\ldots,j_n} \equiv \kappa_n^{k_1,k_2,\ldots,k_n} \left( \sigma^{-1} \right)^{j_1} \left( \sigma^{-1} \right)^{j_2} \cdots \left( \sigma^{-1} \right)^{j_n}
\]

**Hermite Polynomial**

\[
(H_n(\xi))_{j_1,j_2,\ldots,j_n} = (-1)^n e^{\frac{\xi^j}{2}} \frac{\xi^j}{d\xi_{j_1} d\xi_{j_2} \cdots d\xi_{j_n}} e^{-\frac{\xi^j}{2}}
\]

**Short-hand**

\[
h_3(\xi) = \frac{\lambda_3^{j_1,j_2,j_3}}{3!} (H_3(\xi))_{j_1,j_2,j_3}
\]

**Final Line**

\[
Z^Q_j \simeq Z^{GCE} \frac{e^{-\frac{\xi^j}{2}}}{(2\pi V)^J/2 \det |\sigma|} \left[ 1 + \frac{h_3(\xi)}{\sqrt{V}} + \frac{h_4(\xi)}{V} + \frac{h_5(\xi)}{V^{3/2}} + \mathcal{O} \left( V^{-2} \right) \right]
\]

M.H. MSc Thesis, University of Cape Town
Quality of Approximation

Large Volume Limit
- Multiplicity Distribution becomes Gaussian (CLT)
- \( \mu \rightarrow \mu_{GCE} \)
- \( T \rightarrow T_{GCE} \)

Finite Volume Correction
- Given by different order of Gram-Charlier expansion (GC3 - GC5)
- In general only applicable to ‘body‘ of distribution
- Very good description, even for small system size

M.H., V.V.Begun, M.I. Gorenstein, arXiv:0706.3290 [nucl-th]
Mean Values

Equivalence of ensembles holds for mean values in the thermodynamic limit.

However .......

This seems not to apply to higher moments of a distribution!

Express the Width by:

**Scaled Variance**

\[ \omega = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} \]

**Asymptotic Behavior**

\[ V, E \to \infty, \text{ and } E/V = \text{const} \]

\[ \omega_{gce} = 1, \text{ but } \omega_{mce} \to 0.25 \]

For instance:
J. Cleymans and K. Redlich,
J. Cleymans, H. Oeschler, K. Redlich, and S. Wheaton,
F. Becattini, J. Manninen, and M. Gaździcki,
A. Andronic, P. Braun-Munzinger, J. Stachel,

Hadron Resonance Gas Model
- Used as an effective model of strong interaction
- Includes all hadrons and resonances up to $\sim 2$ GeV
- Depending on the version it assumes partial or complete chemical equilibrium
- However, thermal equilibrium is always assumed
Despite its simplicity, it fits a broad range of data:

- with just 4 parameters
- interpretation, however, remains ‘controversial’

Scaled Variance, Full Acceptance

The arrows:
indicate the effect of resonance decay

MCE
Energy conservation leads to correlation with (neutral) particles, thus suppresses final state fluctuations.

V.V. Begun, M. Gaździcki, M.I. Gorenstein, M. H., V.P. Konchakovski, and B. Lungwitz,
Scaled Variance Measured by NA49

Thick Lines

Acceptance Scaling

$$\omega^{\text{acc}} = 1 - q + q\omega^{4\pi}$$

Grey Area

Guess-timate of effect of acceptance in momentum space in MCE

B. Lungwitz et al. [NA49 Collaboration], PoS C FRNC2006 (2006) 024
M.H., arXiv:0710.3938 [nucl-th]
Comparison to NA49 Fluctuation Data

20 AGeV  30 AGeV  40 AGeV  80 AGeV  158 AGeV

B. Lungwitz et al. [NA49 Collaboration], PoS C FRNC2006 (2006) 024
V.V. Begun, M. Gaździcki, M.I. Gorenstein, M. H., V.P. Konchakovski, and B. Lungwitz,
Momentum Space Dependence of Multiplicity Fluctuations in Models

Boltzmann pion gas at $T = 160\,\text{MeV}$ and zero charge density.

Each bin contains same fraction of total yield
Bars indicate size of the bin

Energy and momentum conservation lead to suppressed multiplicity fluctuations at high $|y|$ and $p_T$.

M.H., arXiv:0710.3938 [nucl-th]
Momentum Space Dependence of Multiplicity Fluctuations

Boltzmann pion gas at $T = 160\text{MeV}$ and zero charge density.

Each bin contains same fraction of total yield
Bars indicate size of the bin

Energy and momentum conservation lead to suppressed multiplicity fluctuations at high $|y|$ and $p_T$.

M.H., arXiv:0710.3938 [nucl-th]
Momentum Cuts in UrQMD

UrQMD simulation of central Pb+Pb collision at $b=0$

Construction of bins is the same as before.

MCE suppression of fluctuations also in non-equilibrium systems?

Momentum Space Dependence of Multiplicity Fluctuations in Data

Momentum Cuts in NA49 Data

UrQMD vs. NA49 158AGeV Pb+Pb data

Rapidity and transverse momentum dependence also seen in data!

MCE effects are of similar magnitude as proposed enhancement due to a phase transition / critical point!

B. Lungwitz et al. [NA49 Collaboration], arXiv:0709.1646 [nucl-ex]
Asymptotic Distributions

GCE Energy Distribution

\[ P_{\text{gce}}(E) = \frac{1}{\sqrt{2\pi V\sigma_E^2}} \exp \left[ -\frac{1}{2} \frac{(E - \langle E \rangle)^2}{V\sigma_E^2} \right] \]

GCE Energy and Particle Number Distribution

\[ P_{\text{gce}}(E, N) = \frac{1}{2\pi V \sqrt{\sigma_E^2 \sigma_N^2 (1-\rho^2)}} \exp \left[ -\frac{1}{2V} \left( \frac{(E - \langle E \rangle)^2}{\sigma_E^2 (1-\rho^2)} - 2\rho \frac{(E - \langle E \rangle)(N - \langle N \rangle)}{\sigma_E \sigma_N (1-\rho^2)} + \frac{(N - \langle N \rangle)^2}{\sigma_N^2 (1-\rho^2)} \right) \right] \]

MCE Particle Number Distribution

\[ P_{\text{mce}}(N) = P_{\text{gce}}(N|E) = \frac{P_{\text{gce}}(E,N)}{P_{\text{gce}}(E)} \]

GCE Particle Number Distribution

\[ P_{\text{gce}}(N) = \int dE \ P_{\text{gce}}(E) P_{\text{gce}}(N|E) \]
A More General Energy Distribution

Gaussian Energy Distribution

\[ P_\alpha(E) = \frac{1}{\sqrt{2\pi V\sigma^2_E}} \alpha^2 \exp \left[ -\frac{1}{2} \frac{(E-\langle E \rangle)^2}{V\sigma^2_E} \right] \]

\[ V = 1000 \text{ fm}^3 \quad T = 160 \text{MeV} \]
\[ \lim_{\alpha \to 0} P_\alpha(N) = P_{mce}(N) \]
$0 < \alpha < 1 \implies \text{Larger than MCE Multiplicity Fluctuations}$
\[ \alpha = 1 \implies P_\alpha(N) = P_{gce}(N) \]
\[ \alpha > 1 \quad \implies \quad \text{Anomalous Multiplicity Fluctuations} \]
Fluctuations of Extensive Quantities

\[ \alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations} \]
Fluctuations of Extensive Quantities

\[ \alpha > 1 \] $\Rightarrow$ Anomalous Multiplicity Fluctuations

Conserved Quantities
$\alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations}$

Michael Hauer

Multiplicity Fluctuations in Relativistic Gases

Kraków 16 Nov 07
\[ \alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations} \]
\( \alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations} \)
\[ \alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations} \]
\[ \alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations} \]
\( \alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations} \)
\( \alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations} \)
$\alpha > 1 \Rightarrow$ Anomalous Multiplicity Fluctuations
\( \alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations} \)
$\alpha > 1 \quad \implies \quad \text{Anomalous Multiplicity Fluctuations}$
\[ \alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations} \]
$\alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations}$
$\alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations}$
$\alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations}$
$\alpha > 1 \Rightarrow \text{Anomalous Multiplicity Fluctuations}$
\( \alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations} \)
$\alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations}$
$\alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations}$
$\alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations}$
$\alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations}$
$\alpha > 1 \quad \implies \quad \text{Anomalous Multiplicity Fluctuations}$
$\alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations}$
\[ \alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations} \]
\( \alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations} \)
\[ \alpha > 1 \implies \text{Anomalous Multiplicity Fluctuations} \]
Gaussian Multiplicity Distribution

\[ P_\alpha(N) = \int dE \, P_\alpha(E) \, P_{gce}(N | E) \]

\[ P_\alpha(N) = \frac{1}{\sqrt{2\pi \langle N \rangle \omega_\alpha}} \exp \left[ -\frac{1}{2} \frac{(N - \langle N \rangle)^2}{\langle N \rangle \omega_\alpha} \right] \]

Scaled Variance

\[ \omega_\alpha = \omega_{mce} + \alpha^2 (\omega_{gce} - \omega_{mce}) \]

Qualitative Change in the \( \Delta p_T \) Dependence of \( \omega \) with \( \alpha \)

M.I. Gorenstein, M.H., in preparation
System Size Fluctuations

Gaussian Volume Distribution

$$P_{\eta}(V) = \frac{1}{\sqrt{2\pi \eta^2}} \exp \left[ -\frac{1}{2} \frac{(V - \bar{V})^2}{\eta^2} \right]$$

Multiplicity Distribution

$$P_{\alpha,\eta}(N) = \int dV P_{\eta}(V) P_\alpha(N)$$

Raw Moments

$$\langle N^k \rangle_{\alpha,\eta} = \int dN N^k P_{\alpha,\eta}(N)$$

Scaled Variance

$$\omega_{\eta,\alpha} = \omega_V \langle \rho_N \rangle + \omega_\alpha$$

Some Definitions

- $\bar{V}$: average volume
- $\langle \rho_N \rangle$: particle number density
- $\omega_V = \eta^2 / \bar{V}$

Independent source model type of result!

Michael Hauer  
Multiplicity Fluctuations in Relativistic Gases

Kraków  16 Nov 07  29 / 32
Gaussian Volume Distribution

\[ P_{\bar{\eta}}(V) = \frac{1}{\sqrt{2\pi \eta^2}} \exp \left[ -\frac{1}{2} \left( \frac{V - \bar{V}}{\eta^2} \right)^2 \right] \]

\[ \bar{\eta} = \eta / \bar{V} \]
System Size Fluctuations

Gaussian Volume Distribution

\[ P_{\eta}(V) = \frac{1}{\sqrt{2\pi \eta^2}} \exp \left[ -\frac{1}{2} \frac{(V - \bar{V})^2}{\eta^2} \right] \]

\[ \bar{\eta} = \eta / \bar{V} \]

\[ \bar{\eta}^2 = \omega_V / \bar{V} \]
Multiplicty Distribution

\[ P_{\alpha,\eta}(N) = \int dV \ P_{\eta}(V) \ P_{\alpha}(N) \]

Scaled Variance

\[ \omega_{\eta,\alpha} = \omega_{\eta} \langle \rho_N \rangle + \omega_{\alpha} \]

NO Qualitative Change in the \( \Delta p_T \) Dependence of \( \omega \) with \( \eta \)

M.I. Gorenstein, M.H., in preparation
Particle number fluctuations were discussed in different statistical ensembles and compared to NA49 data on nucleus-nucleus collisions.

Fluctuations are different in different ensembles!

GCE and CE are in clear contradiction to data!

Data, UrQMD as well as MCE show suppressed fluctuations in momentum bins with high $p_T$ and $y$.

Fluctuations are an important test for the statistical hadronization model.
Apart from Conservation Laws

Van der Waals Gas

GCE Van der Waals Gas

Has been used to:

- model repulsive interactions between hadrons
- suppression of densities can be removed by rescaling the system volume
- suppression of fluctuations is qualitatively different
- could be a first step towards a simple model with a phase transition
- can be extended to include conservation laws


‘Quantum pion‘ gas at $T = 160\,\text{MeV}$ and zero charge density.

- Each bin contains same fraction of total yield
- Bars indicate size of the bin

Quantum effects could be visible for Pions in experimental data!
Effect is strongest in low $p_T$ bins.
'Quantum pion' gas at $T = 160\,\text{MeV}$ and zero charge density.

- Each bin contains same fraction of total yield
- Bars indicate size of the bin

Quantum effects could be visible for Pions in experimental data!
Effect is strongest in low $p_T$ bins.