# Multiplicity Fluctuations in Statistical Models

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**Outline**



H-QM | Helmholtz Research School
Quark Matter Studies

FIAS Frankfurt Institute
for Advanced Studies

# Statistical Ensembles

| Grand Canonical | Canonical | Microcanonical |
| --- | --- | --- |
| $V, T, \mu$ | $V, T, Q$ | $V, E, Q$ |
| $\langle E \rangle, \langle Q \rangle, \langle \vec{P} \rangle = \vec{0}$ | $\langle E \rangle, \langle \vec{P} \rangle = \vec{0}$ | $\vec{P} = \vec{0}$ |

Thermodynamic Limit : $\langle N \rangle \rightarrow \infty, V \rightarrow \infty, \frac{\langle N \rangle}{V} = const$

# Why has there been so little interest so far?

## Experiment

Experimentally this means event by event analysis of data

- Very precise determination of collision centrality
- Good understanding of acceptance and resolution

# Why has there been so little interest so far?

## Thermal Model

- Statistical ensembles are equivalent under the thermodynamic limit
- Grand canonical ensemble often sufficient
- Canonical effects only become important when only few particles are produced
- Quantum statistics effects are generally small (10% for pions)

But: all this changes when one is interested in multiplicity fluctuations

# Grand Canonical Boltzmann Pion Gas

Single particle partition function

$$z = \frac{gV}{2\pi^2} \int dp\, p^2 e^{-\frac{E}{T}} = \frac{gV}{2\pi^2} m^2 T\, K_2\left(\frac{m}{T}\right)$$

System partition function

$$Z^{GCE} = \exp\left(z\, e^{\frac{\mu}{T}} \lambda_+ + z\, e^{-\frac{\mu}{T}} \lambda_-\right)$$

Expectation values

$$\langle N_+ \rangle = \frac{1}{Z^{GCE}} \left(\lambda_+ \frac{\partial}{\partial \lambda_+}\right) Z^{GCE}\Big|_{\lambda_\pm = 1} = z\, e^{\frac{\mu}{T}}$$

$$\langle N_+^2 \rangle = \frac{1}{Z^{GCE}} \left(\lambda_+ \frac{\partial}{\partial \lambda_+}\right)^2 Z^{GCE}\Big|_{\lambda_\pm = 1} = \left(z\, e^{\frac{\mu}{T}}\right)^2 + z\, e^{\frac{\mu}{T}}$$

Scaled variance

$$\omega = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} = 1$$ Poisson !

# Canonical Boltzmann Pion Gas

System partition function
$$Z^Q = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{-iQ\phi} \exp\left( z e^{i\phi} \lambda_+ + z e^{-i\phi} \lambda_- \right)\Big|_{\lambda_\pm = 1} = I_Q(2z)$$

Expectation values
$$\langle N_+ \rangle = \frac{1}{Z^Q} \left( \lambda_+ \frac{\partial}{\partial \lambda_+} \right) Z^Q \Big|_{\lambda_\pm = 1} = \frac{I_{Q-1}(2z)}{I_Q(2z)} \; z$$

$$\langle N_+^2 \rangle = \frac{1}{Z^Q} \left( \lambda_+ \frac{\partial}{\partial \lambda_+} \right)^2 Z^Q \Big|_{\lambda_\pm = 1} = \frac{I_{Q-2}(2z)}{I_Q(2z)} \; z^2 + \frac{I_{Q-1}(2z)}{I_Q(2z)} \; z$$

Scaled variance
$$\omega_+ = \frac{\langle N_+^2 \rangle - \langle N_+ \rangle^2}{\langle N_+ \rangle} \xrightarrow[V \to \infty]{} \frac{1}{2}$$

Not 1!

( for a neutral system )

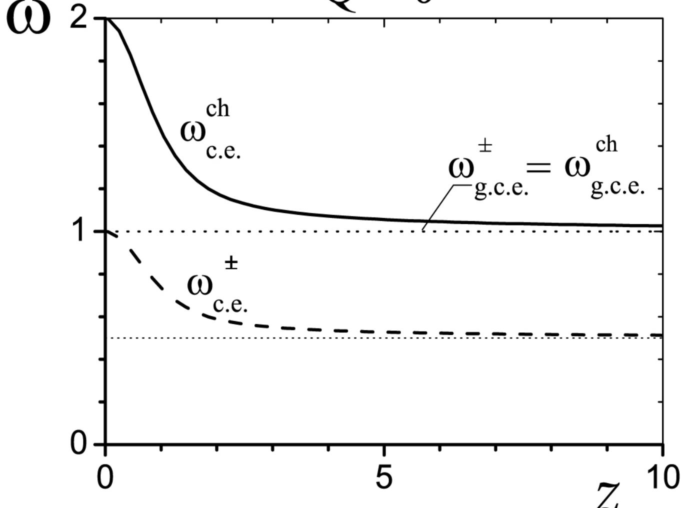
V.Begun, M.Gorenstein, M.Gazdzicki, O.Zozulya, Phys.Rev. C 70 (2004) 034901

# Finite Volume Pion gas

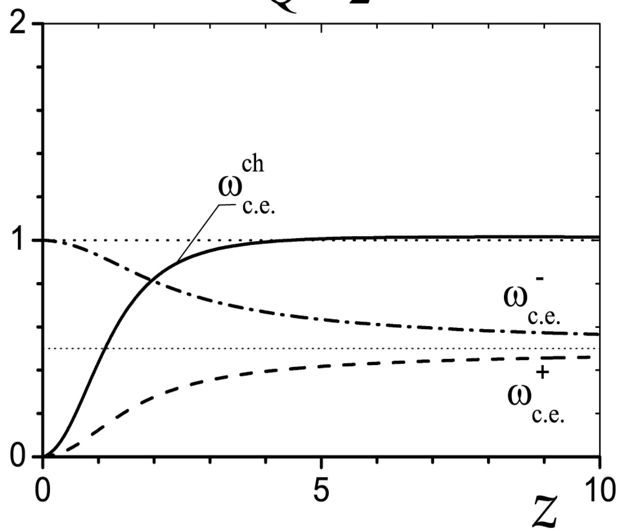
(Boltzmann)

## Canonical Ensemble

Q = 0



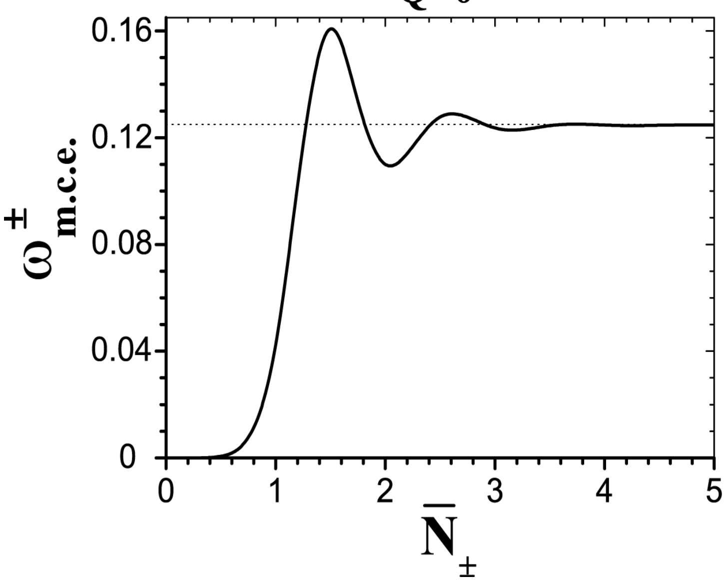

Q = 2




V.Begun, M.Gorenstein, O.Zozulya,
Phys.Rev. C 72 (2005) 014909

## Microcanonical Ensemble
massless gas

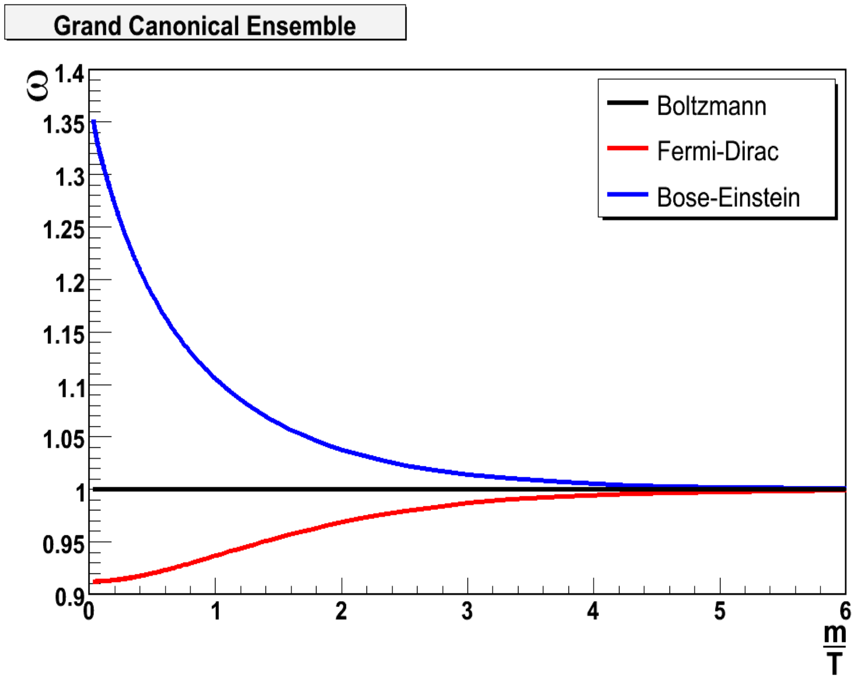
**Q=0**

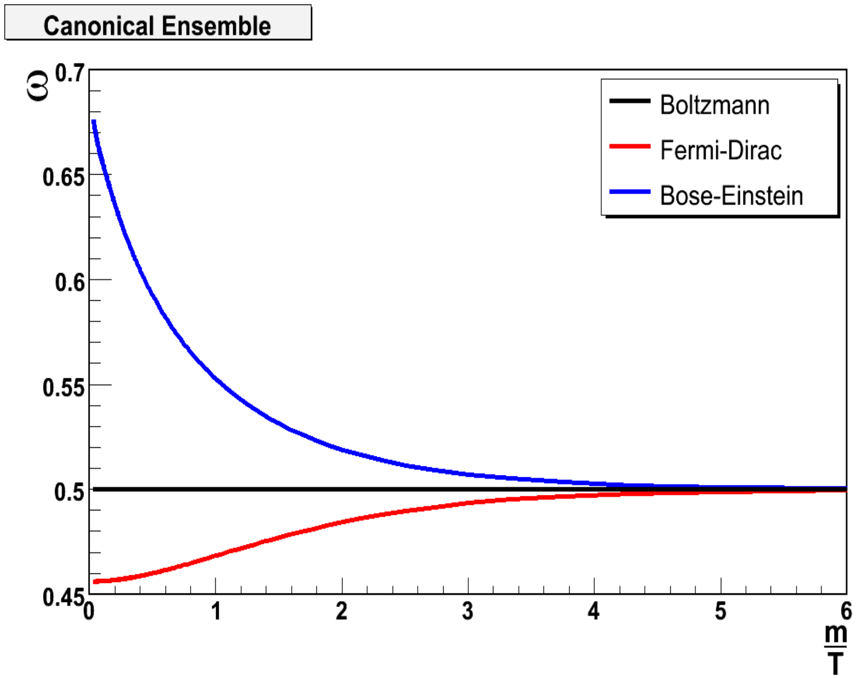


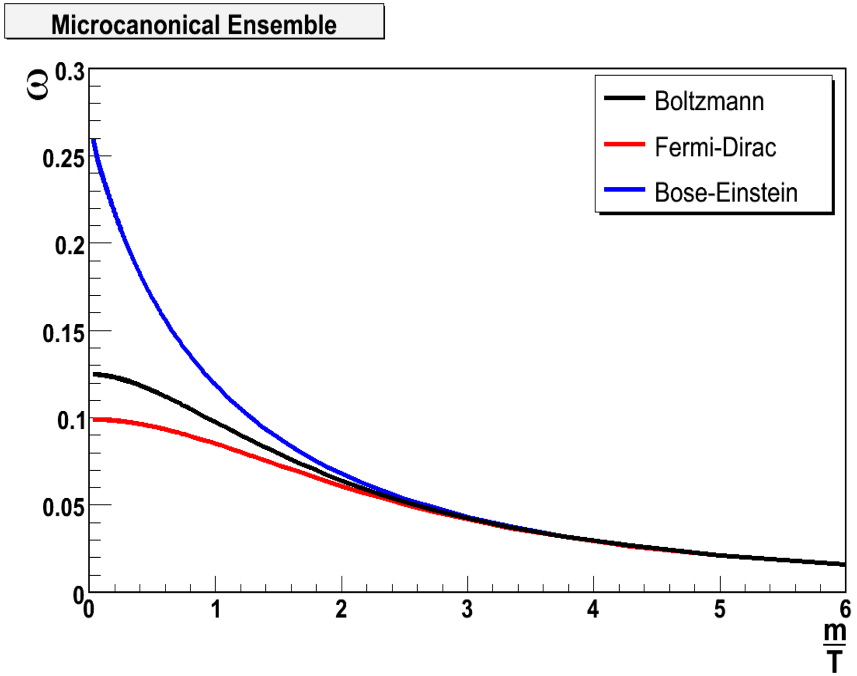

V.Begun, M.Gorenstein,
A.Kostyuk, O. Zozulya
Phys.Rev. C 71 (2005) 054904

Very fast convergence to asymptotic values for a neutral sytem !

# Thermodynamic Limit Pion Gas



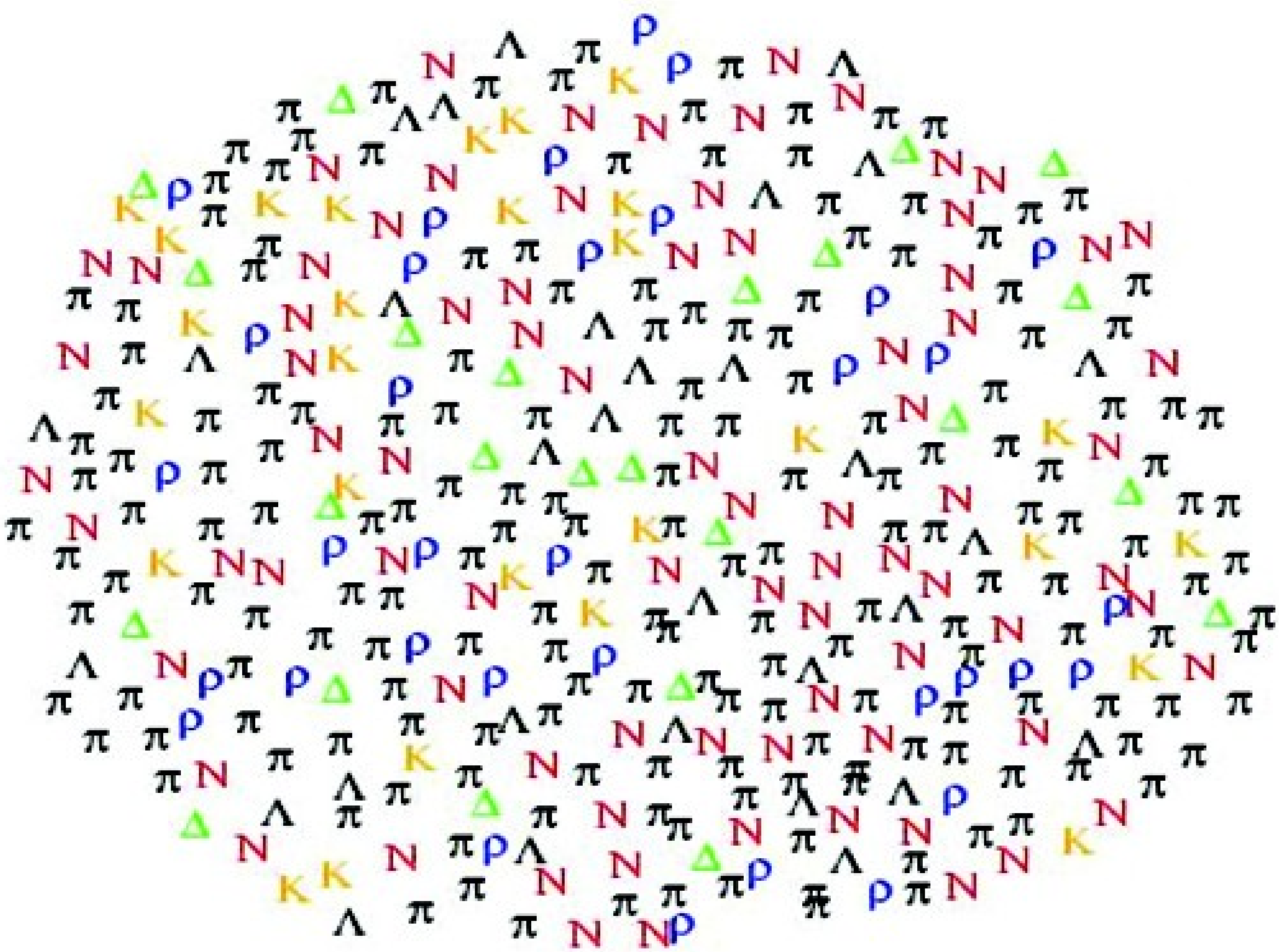

$\pi^+ \quad \pi^-$ Gas

$Q=0 \rightarrow \mu=0 \quad \Rightarrow \quad \omega^+ = \omega^-$

Quantum effects can be quite large, even in a neutral system!

V.Begun, M.Gorenstein, A.Kostyuk, O.Zozulya
nucl-th/0505069

# Hadron Resonance Gas



# Hadron Resonance Gas

## Canonical Partition Function

$$Z^{Q^j} = \left[ \prod_{j=1}^{3} \int_{-\pi}^{\pi} \frac{d\phi_j}{2\pi} e^{-iQ^j \phi_j} \right] Z^{GC}(\{\lambda_l\})$$

$$\langle N_k^n \rangle = \frac{1}{Z^{Q^j}} \left( \lambda_k \frac{\partial}{\partial \lambda_k} \right)^n Z^{Q^j}$$

$$Z^{GC}(\{\lambda_l\}) = \exp\left[ \sum_l z_l(\lambda_l) \right]$$

$$\lambda_l = \exp\left( i \sum_{j=1}^{3} q_l^j \phi_j \right)$$

$$z_l(\lambda_l) = \frac{g_l V}{2\pi^2} \int_0^{\infty} p^2 \, dp \ln\left( 1 \pm \exp\left[ -\frac{E_l}{T} \right] \lambda_l \right)^{\pm 1}$$

$$q_l^j = (b_l, s_l, q_l)$$

- No practical analytical solution is known
- Only in Boltzmann approximation is an analytical reduction of integrals possible
- heavily oscilating integrand makes numerical evaluation expensive



A.Keränen, F.Becattini, Phys.Rev. C 65 (2002) 044901

# Central Limit Theorem Expansion

the state we are interested in :

$$P(E, \vec{Q}, \vec{P}, N) = \frac{\text{number of all states with } E, \vec{Q}, \vec{P}, \text{ and } N}{\text{number of all states}}$$

with normalization :

$$P(E, \vec{Q}, \vec{P}) = \frac{\text{number of all states with } E, \vec{Q}, \vec{P}}{\text{number of all states}}$$

and finally our distribution :

$$P_{E, \vec{Q}, \vec{P}}(N) = \frac{P(E, \vec{Q}, \vec{P}, N)}{P(E, \vec{Q}, \vec{P})}$$

# Central Limit Theorem Expansion

$$Z^{Q^j} \approx \prod_{j=1}^{3} \left[ \int_{-\pi}^{\pi} \frac{d\phi_j}{2\pi} \, e^{-i Q^j \phi_j} \right] \exp\left[ V \sum_{n=0}^{\infty} \frac{\kappa_n^{j_1 \dots j_n}}{n!} \phi_{j_1} \dots \phi_{j_n} \right]$$

Cumulant tensor $\kappa_n^{j_1 \dots j_n} = \left(-i\right)^n \frac{\partial^n \Psi\left(\vec{\phi}\right)}{\partial \phi_{j_1} \dots \partial \phi_{j_n}}$

$\Psi\left(\vec{\phi}\right) = \sum_l \frac{z_l\left(\vec{\phi}\right)}{V}$ Generating function

3-dim Gaussian with first correction term $O\left(V^{-1/2}\right)$

$$Z^{Q^j} \simeq \frac{Z^{GCE}}{\left(2\pi V\right)^{2/3} det\,\sigma} \exp\left[\frac{-\xi^j \xi_j}{2}\right] \left(1 + O\left(V^{-1/2}\right)\right)$$

where $\sigma \equiv \kappa_2^{1/2}$ ,and $\xi^j = \left(Q^k - V \kappa_1^k\right) \left(\sigma^{-1}\right)_k^{\;j} V^{-1/2}$

Find physical fugacities from $\frac{\partial Z^{\vec{Q}}}{\partial \vec{Q}} = \vec{0}$

$$\left(\kappa_1^B, \kappa_1^S, \kappa_1^Q\right) \underset{V \to \infty}{\rightarrow} \left(\rho_B, \rho_S, \rho_Q\right)$$

# Central Limit Theorem Expansion

Canonical Normalization
$$P\left(V\kappa_1^j\right) = \frac{Z^{V\kappa_1^j}}{Z^{GC}} \simeq \frac{1}{\left(2\pi V\right)^{3/2}\, det\,\sigma}$$

4-dim Gaussian
$$P\left(Q^j, N_k\right) = \frac{Z^{Q^j, N_k}}{Z^{GC}} \simeq \frac{1}{\left(2\pi V\right)^{4/2}\, det\,\tilde{\sigma}} \exp\left[-\frac{\tilde{\xi}^j\ \tilde{\xi}_j}{2}\right]$$

1-dim Gaussian
$$P_{V\kappa_1^j}\left(N_k\right) = \frac{Z^{V\kappa_1^j, N_k}}{Z^{V\kappa_1^j}} \simeq \frac{det\,\sigma}{\left(2\pi V\right)^{1/2}\, det\,\tilde{\sigma}} \exp\left[-\frac{\tilde{\xi}^j\ \tilde{\xi}_j}{2}\right]$$

With variance
$$D^2 = \frac{V\left(det\,\tilde{\sigma}\right)^2}{\left(det\,\sigma\right)^2}$$

And scaled variance
$$\omega_k = \frac{\left(det\,\tilde{\sigma}\right)^2}{\left(det\,\sigma\right)^2 \rho_k} = \frac{det\,\tilde{\kappa}_2}{det\,\kappa_2\ \ \kappa_1^k}$$

Finite volume corrections can be done by Gram Charlier Expansion

# Particle Decay

## Central Limit Theorem Expansion

$$z_l = \frac{g_l}{2\pi^2}\int p^2 dp \ln\left(1 \pm e^{-\beta(E_l-\mu_l)} e^{i q_l^j \phi_j}\left[\sum_{n_k=0}^{C_{l,k}} \Gamma_{l,q}^{n_k} e^{i n_k \phi_k}\right]\right)^{\pm 1}$$

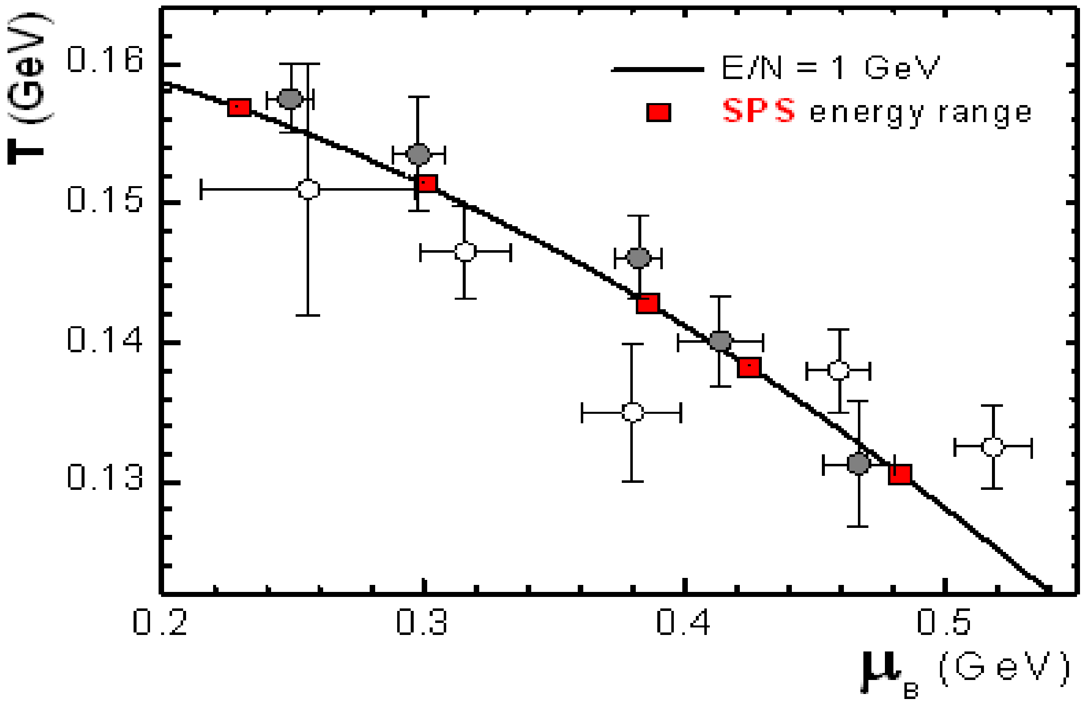
$$\Gamma_{l,q}^{n_k} = \sum_{c_k=n_k}^{C_{l,k}} q^{n_k}(1-q)^{c_k-n_k}\binom{c_k}{n_k}\Gamma_l^{c_k}$$

$$\sum_{n_k=0}^{C_{l,k}} \Gamma_{l,q}^{n_k} = 1$$

| | |
|---|---|
| $q$ | Detection Probability |
| $c_k$ | multiplicity of particle k in this channel |
| $\Gamma_l^{c_k}$ | branching ratio  of resonance l into c_k particles k |
| $n_k$ | number of detected particles |
| $\Gamma_{j,q}^{n_k}$ | corrected branching ratio |

# Chemical Freeze-Out Line

constant average energy per partilce




$$\frac{\langle E \rangle}{\langle N \rangle} \sim 1\text{GeV} \qquad Pb-Pb$$

$$\rho_B \Rightarrow \mu_B$$

$$\rho_S \Rightarrow \mu_S \qquad \rho_S = 0$$

$$\rho_Q \Rightarrow \mu_Q \qquad \rho_Q = 0.4\,\rho_B$$

$$\gamma_S \simeq 1-0.396 \ \exp\left(-1.23 \ \frac{T}{\mu_B}\right)$$

$$\mu_B\left(\sqrt{S_{NN}}\right) \simeq \frac{1.27\,GeV}{1+\sqrt{S_{NN}}/4.3\text{GeV}}$$

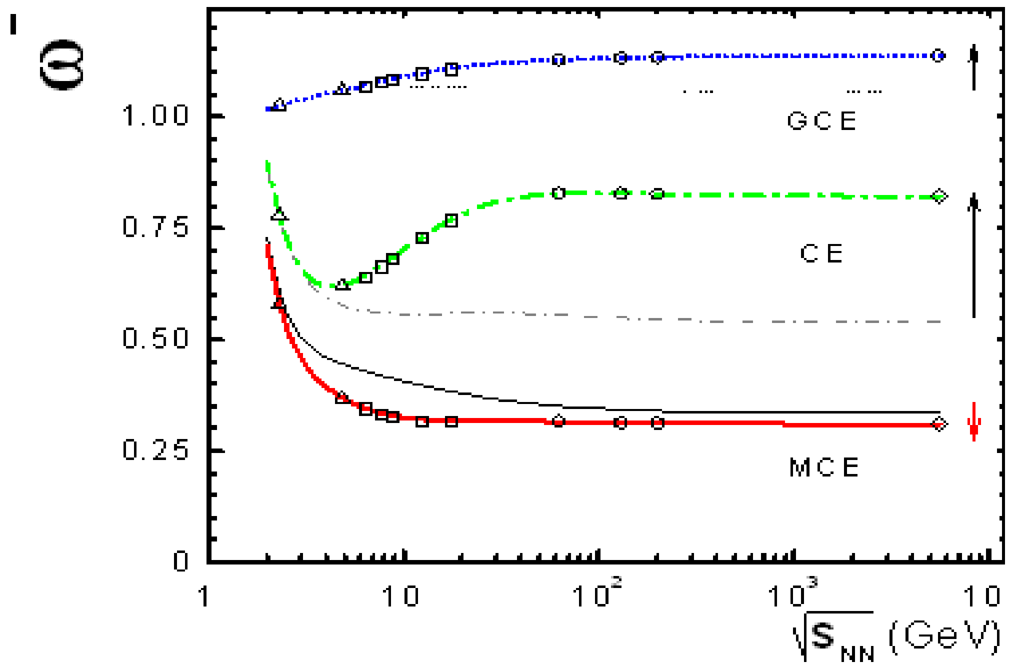
J.Cleymans, K.Redlich
Phys.Rev.Lett. 81 (1998) 5284-5286

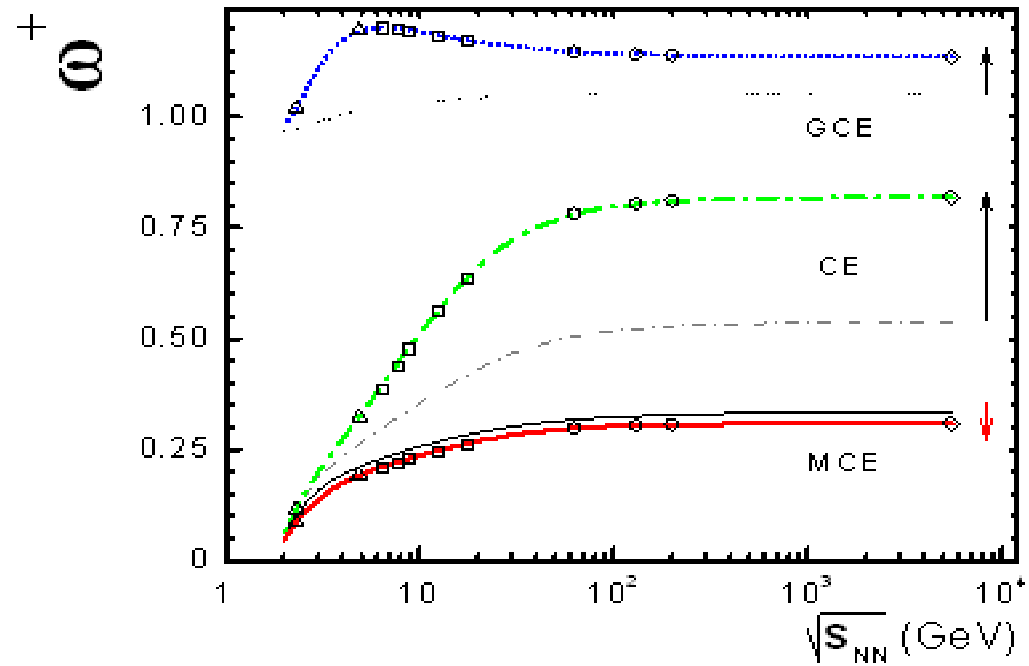
F.Becattini, J.Manninen
Phys.Rev. C73 (2006) 044905

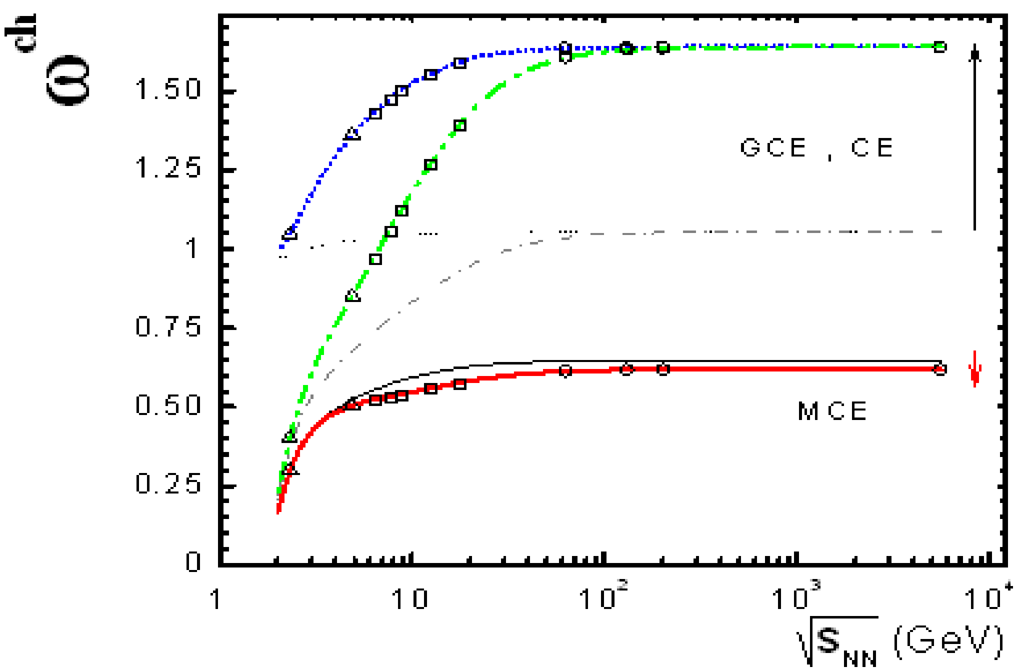
J.Cleymans, H.Oeschler, K.Redlich, S.Wheaton
Phys.Rev. C73 (2006) 034905

A.Andronic, P.Braun-Munzinger, J.Stachel
Phys.Rev. A772 (2006) 167-199

# Scaled Variance



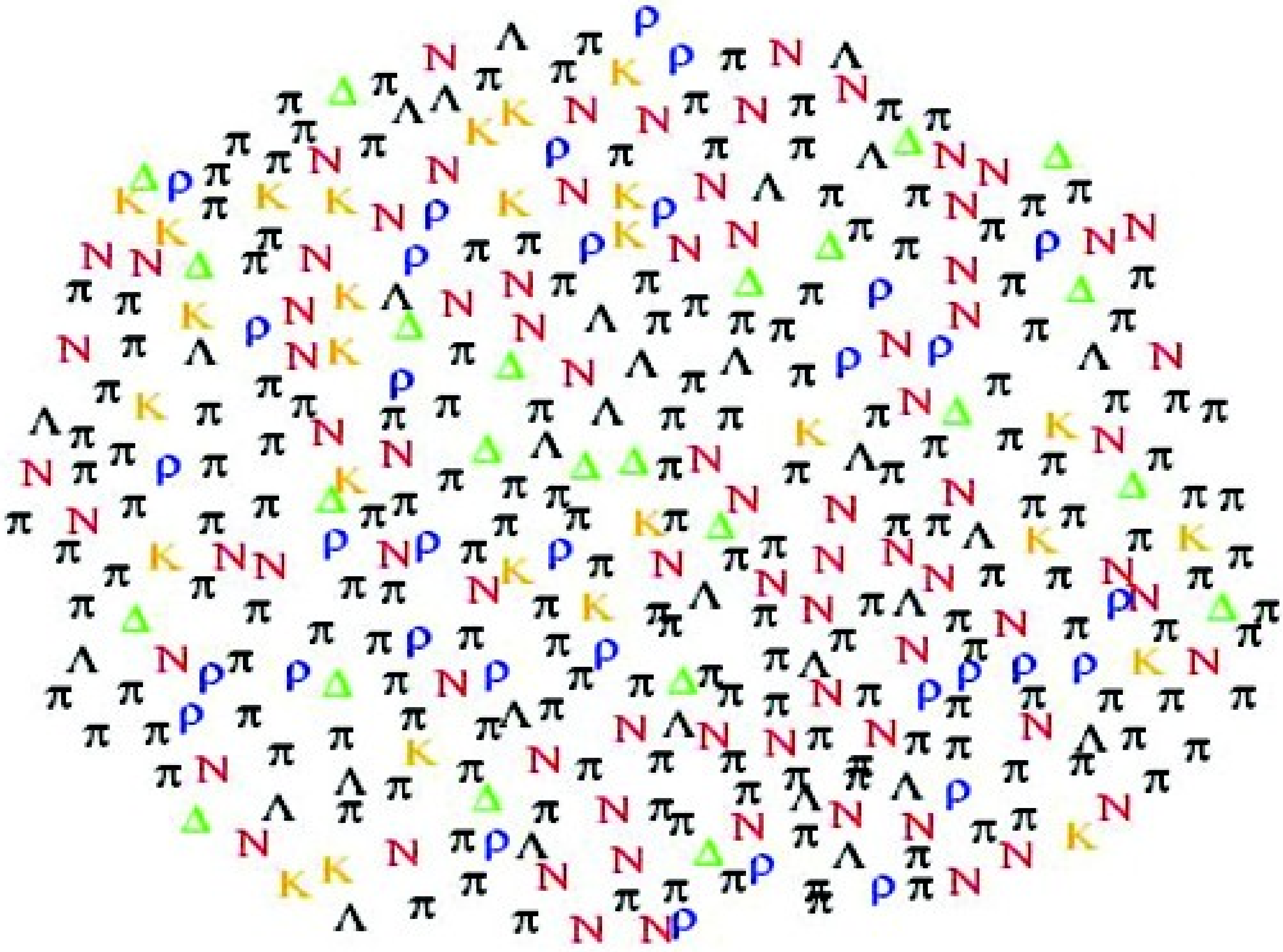




V.Begun, M.Gorenstein, M.H., V.Konchakovski, O.Zozulya Phys.Rev. C74 (2006) 044903

V.Begun, M.Gazdzicki, M.Gorenstein, M.H., V.Konchakovski, B.Lungwitz nucl-th/0611075

# Hadron Resonance Gas

(again)



# Acceptance Scaling

„Uncorrelated detection“

$$P_{acc}(n, N) = q^n (1-q)^{N-n} \frac{N!}{n!(N-n)!}$$

Distribution of detected particles

$$P(n) = \sum_{N=n}^{\infty} P_{4\pi}(N) P_{acc}(n, N)$$

with scaled variance

$$\omega_{acc} = 1 - q + q\,\omega_{4\pi}$$

This would be exact only in (Boltzmann) GCE and CE.

Valid for any type of distribution.

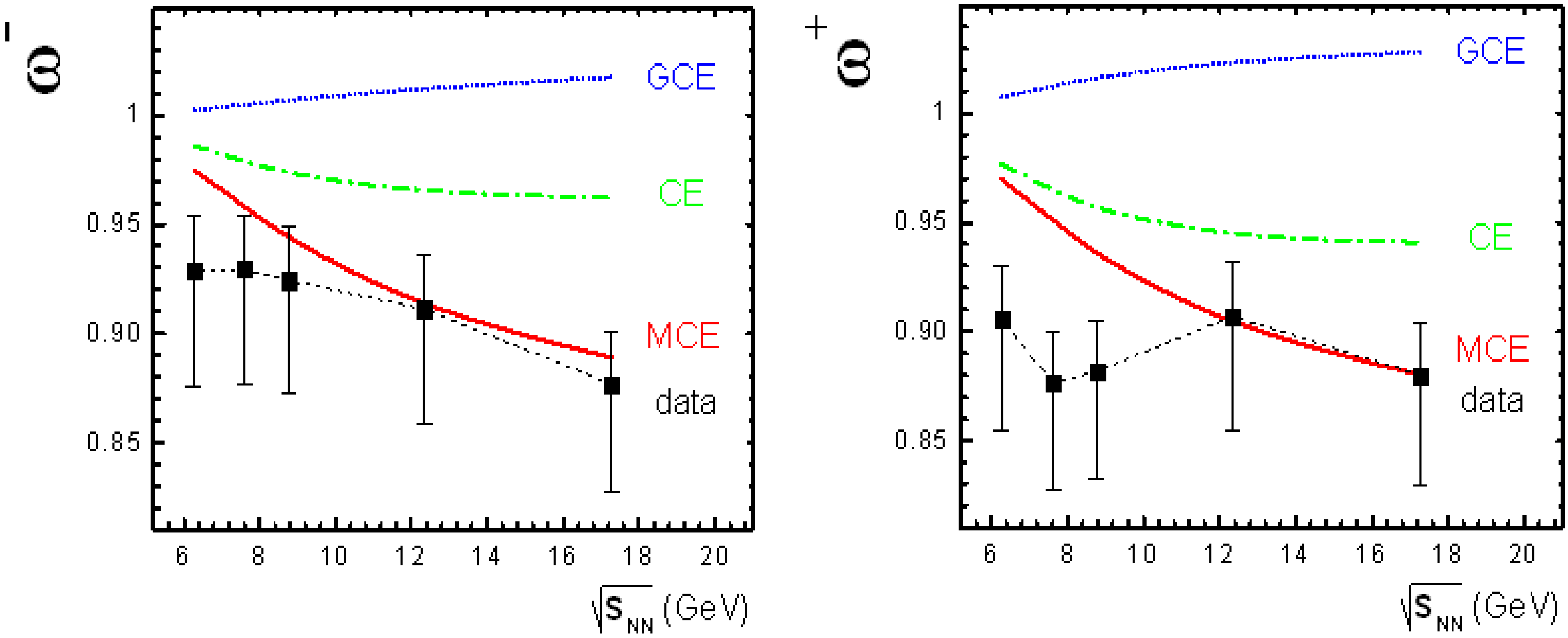
Transfroms Poissonians into Poissonians.

Certainly only a approximation, if particles are correlated in momentum space.

# Comparison with NA49-Data

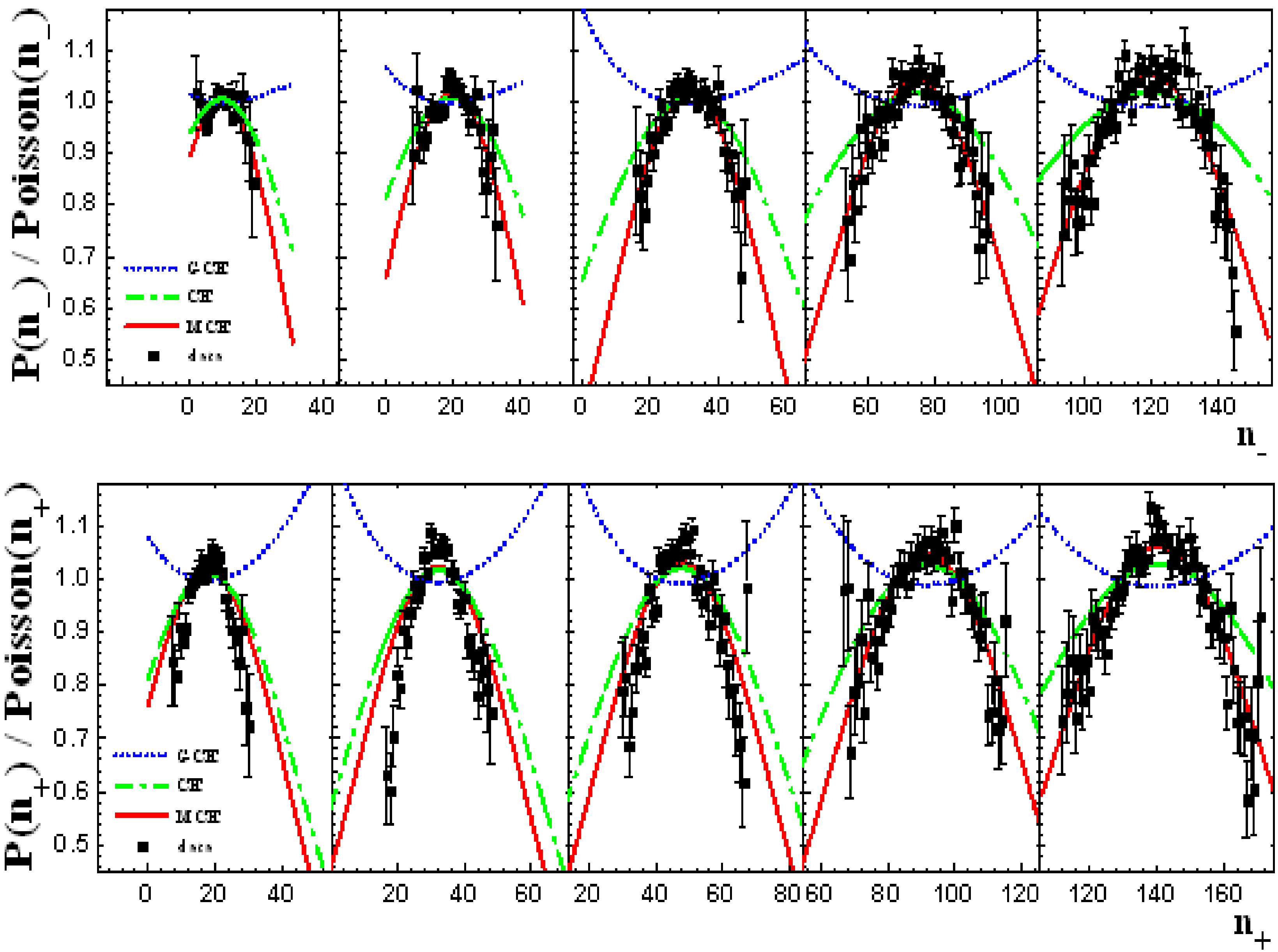
negatively charged hadrons

positively charged hadrons




V.Begun, M.Gazdzicki, M.Gorenstein, M.H., V.Konchakovski, B.Lungwitz
nucl-th/0611075

# Comparison with NA49-Data



# Summary and Outlook

Data is well described by our MCE model !

However :

There should be energy fluctuations !

How good is acceptance scaling in MCE ?

How would flow change our results ?

What about chemical non-equillibrium ?

Phase transition ?

What about clusters ?

...

# Finite Volume Hadron Gas

$T \approx 160\, MeV$
$\gamma_S = 1.0$

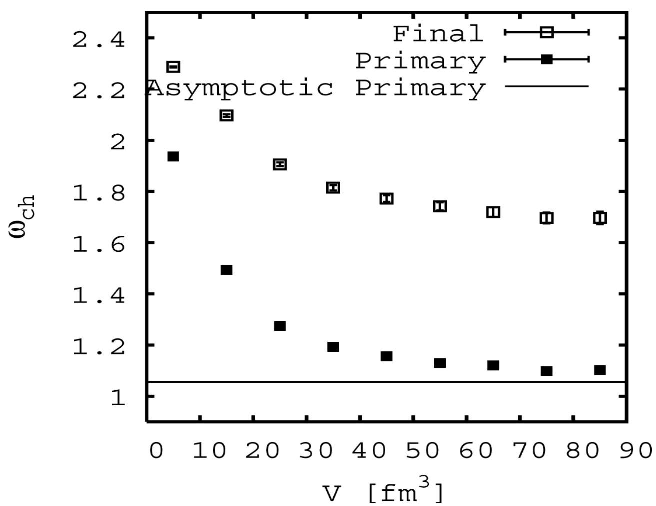
$V = 90\text{fm}^3 \Rightarrow r \approx 2.8\text{fm}$

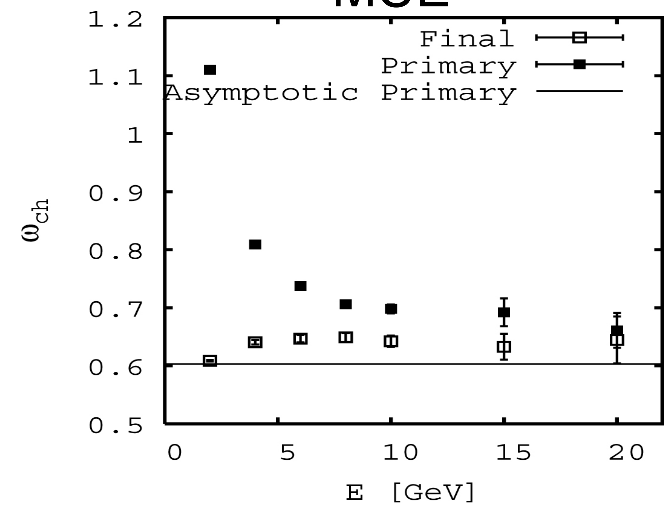
$E = 20\text{GeV}\,(\rho_E = 0.4\text{GeV}) \Rightarrow r \approx 2.3\ fm$

CE

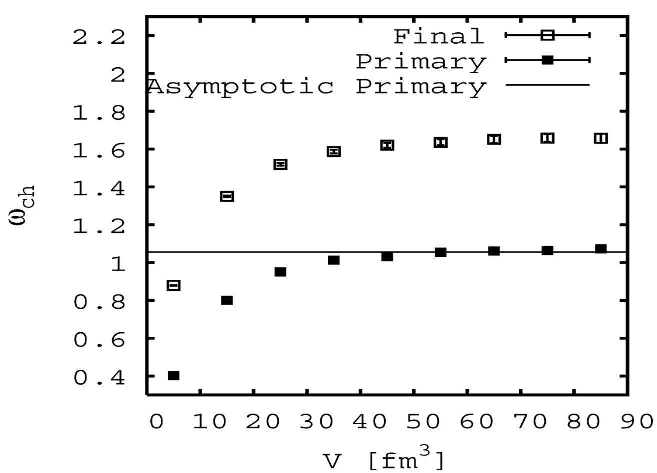
MCE

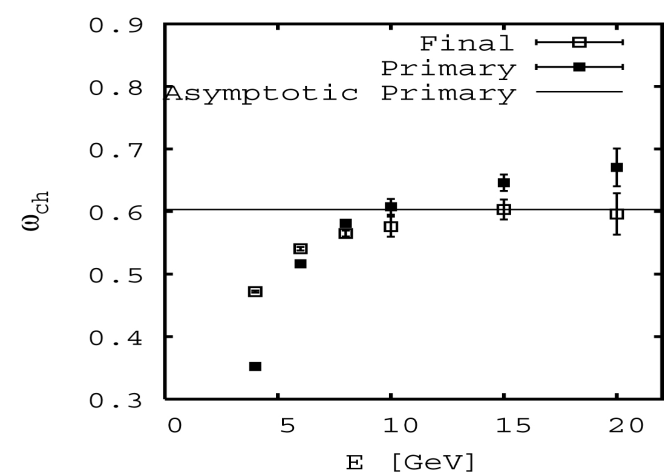
$\{B, S, Q\} = \{0,0,0\}$




$\{B, S, Q\} = \{2,0,2\}$



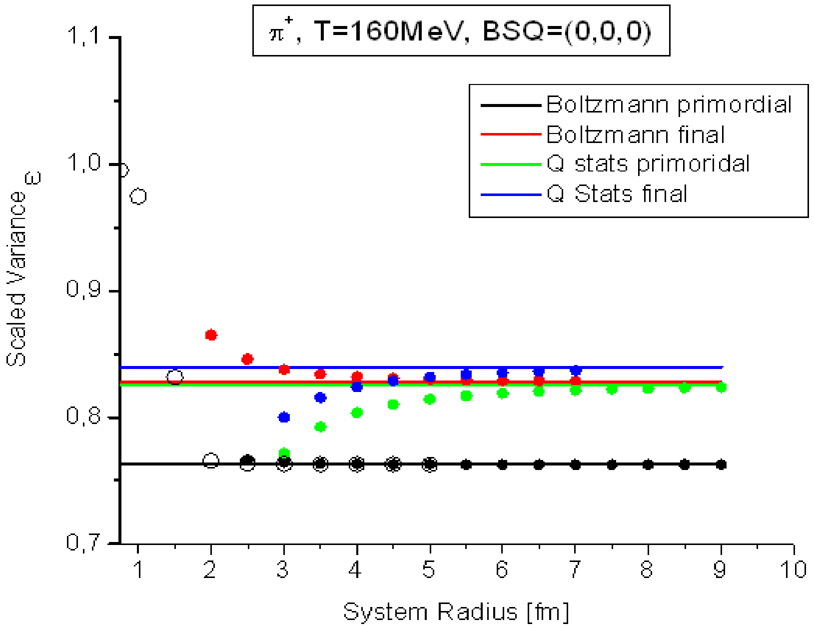



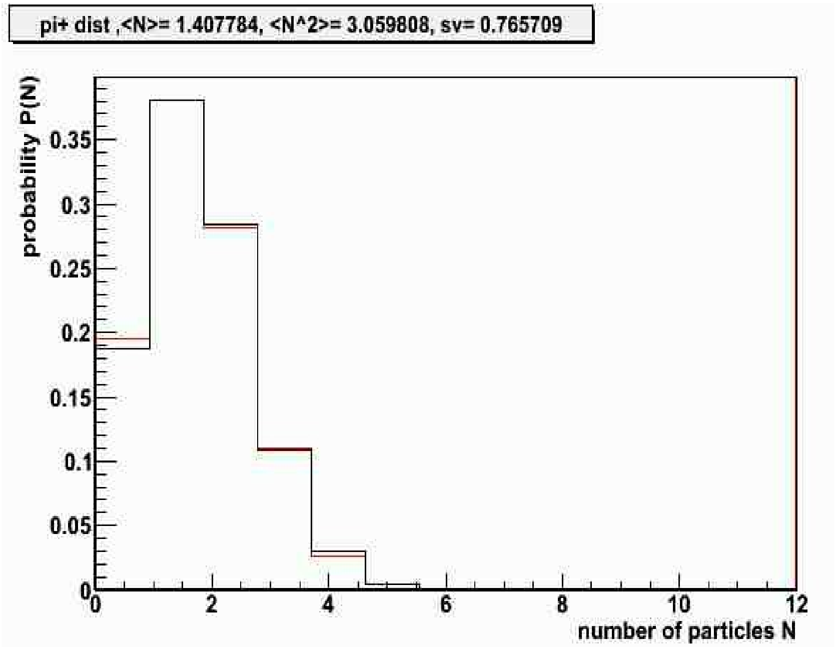



F.Becattini, L.Ferroni, T.Gabbriellini, Phys.Rev. C 65 (2002) 044901

# Finite Volume Corrections

## Gram Charlier Expansion




$$P\left(N_j\right) = \frac{z_j^{N_j}}{N_j!} \frac{Z_{j\ excl}^{\vec{Q}-N_j\vec{Q}_j}}{Z^{\vec{Q}}}$$

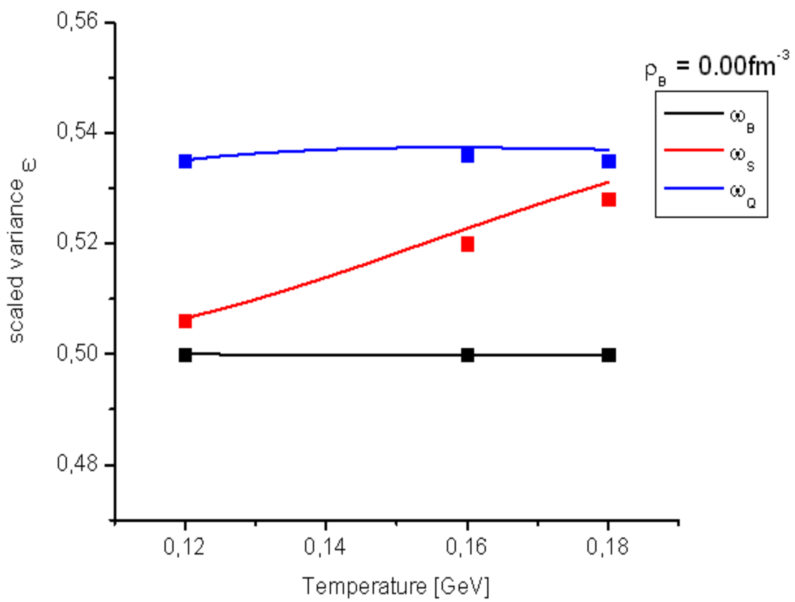
**Primordial Boltzmann**

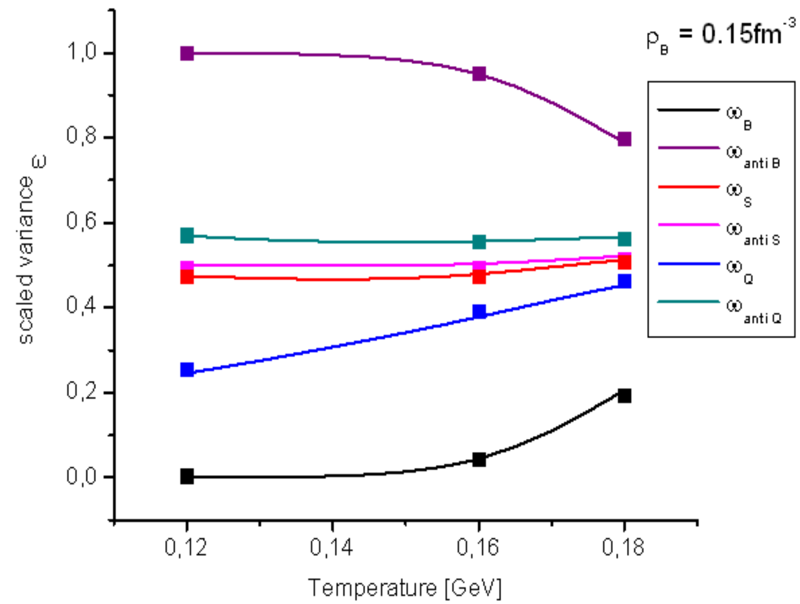
$T = 0.160\text{GeV}$ $\quad\quad \gamma_S = 1.0$

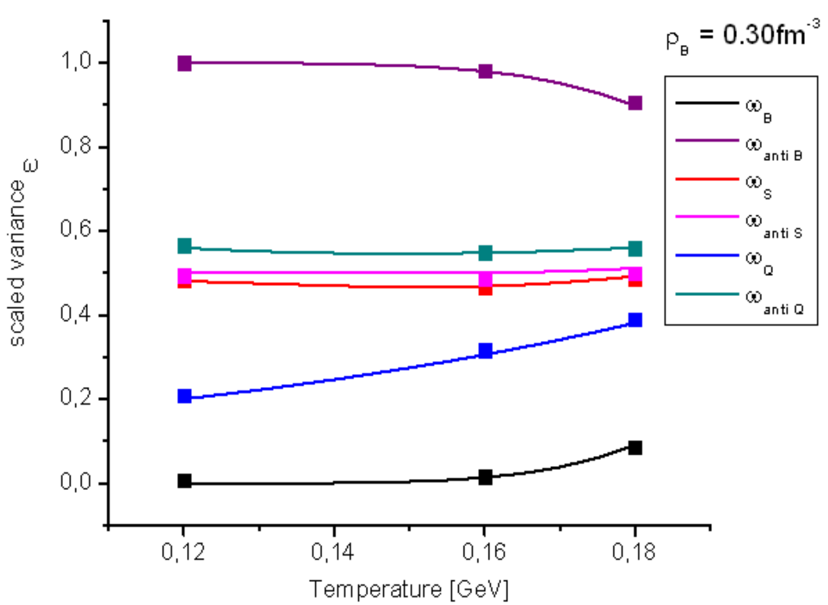
$\left(B, S, Q\right) = \left(0,0,0\right)$ $\quad\quad r = 2.0\ fm$

| $\pi^+$ | *approx* | *exact* |
| --- | --- | --- |
| *Norm* | 0.9954 | 1.0 |
| $\langle N \rangle$ | 1.420 | 1.408 |
| $\omega$ | 0.7667 | 0.7657 |

# Comparison of Methods




Comparison is half hearted, since different particle tables were used!

Microscopic correlator method and Central limit theorem expansion agree „on the dot“. (same table!)



F.Becattini, L.Ferroni, T.Gabbriellini, Phys.Rev. C 72 (2005) 064904

# Techniques

## Microscopic correlaor (GCE and CE)

$$\langle n_{p,k}\rangle = \frac{1}{\exp\left[\beta\left(E_k-\mu_k\right)\right]-\gamma_k}$$

$$\left(\Delta p\right)^3 \frac{gV}{\left(2\pi\right)^3} \gg 1$$

$$v^2_{p,k} = \langle \Delta n^2_{p,k}\rangle = \langle\left(n_{p,k}-\langle n_{p,k}\rangle\right)\rangle = \langle n_{p,k}\rangle\left(1+\gamma_k\langle n_{p,k}\rangle\right)$$

$\gamma_k$ {
+1 Bosons
-1 Fermions
0 Boltzmann

$$\langle \Delta n_{p,k}\Delta n_{q,l}\rangle = \underbrace{v^2_{p,k}\delta_{p,q}\delta_{k,l}}_{GCE\ correlator} - v^2_{p,k}v^2_{q,l}\frac{q_k q_l}{\sum_{p,k} v^2_{p,k}q^2_k}$$

$$\mu_k = q_k\mu \qquad E_k = \sqrt{p^2+m^2_k}$$

In a canonical ensemble the variation needs to vanish $\Delta Q = \sum_{p,k} q_k \Delta n_{p,k} = 0$

$$\langle\left(\Delta N_k\right)^2\rangle = \sum_{p,q,l}\langle \Delta n_{p,k}\Delta n_{q,l}\rangle$$

$$\langle N_k\rangle = \sum_p \langle n_{p,k}\rangle$$

→

$$\omega_k \equiv \frac{\langle\left(\Delta N_k\right)^2\rangle}{\langle N_k\rangle}$$

V.Begun, M.Gorenstein, O.Zozulya, Phys.Rev. C 72 (2005) 014909

# Particle Decay

## Microscopic Correlator

$$G \equiv \prod_R \left( \sum_r b_r^R \prod_i \lambda_i^{n_{i,r}} \right)^{N_R}$$

$$\langle N_i^k \rangle_R = \left[ \lambda_i \frac{\partial}{\partial \lambda_i} \right]^k G$$

| | |
|---|---|
| $b_r^R$ | Branching ration of channel r of resonance R |
| $n_{i,r}$ | multiplicity if species i in channel r |
| $\langle n_i \rangle_R$ | average multplicity of i from decay of R |
| $\langle N_R \rangle$ | average primordial density of R |

$$\langle \Delta N_i \Delta N_j \rangle = \langle \Delta N_i^* \Delta N_j^* \rangle + \sum_R \left[ \langle \Delta N_R^2 \rangle \langle n_i \rangle_R \langle n_j \rangle_R + \langle N_R \rangle \langle \Delta n_i \Delta n_j \rangle_R \right]$$

GCE

$$\langle \Delta N_i \Delta N_j \rangle = \langle \Delta N_i^* \Delta N_j^* \rangle + \sum_R \left[ \langle \Delta N_R^2 \rangle \langle n_i \rangle_R \langle n_j \rangle_R + \langle N_R \rangle \langle \Delta n_i \Delta n_j \rangle_R \right]$$
$$+ \sum_R \langle \Delta N_i^* \Delta N_R \rangle \langle n_j \rangle_R + \sum_R \langle \Delta N_j^* \Delta N_R \rangle \langle n_i \rangle_R + \sum_{R \neq R'} \langle \Delta N_R \Delta N_{R'} \rangle \langle n_i \rangle_R \langle n_j \rangle_{R'}$$

CE

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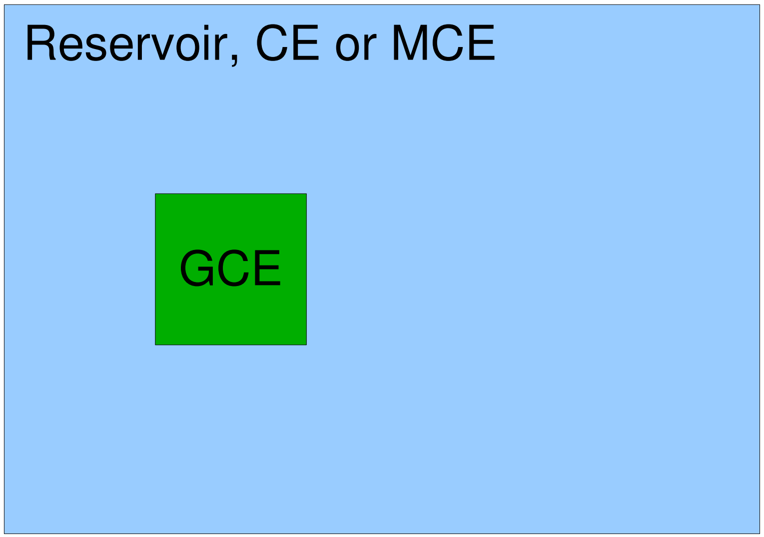
# Pseudo intensive quantities




GCE is defined as a small subsystem of a large reservoir

$$D^2 = \langle N^2 \rangle - \langle N \rangle^2 \neq \sum_{k=1}^{N} \langle N_k^2 \rangle - \langle N_k \rangle^2$$

Variance is an extensive quantity, but not additive

$$D^2 = \langle (\Delta N)^2 \rangle = \sum_{k=1}^{M} \sum_{l=1}^{M} \langle \Delta N_k \Delta N_l \rangle$$

Correlators vanish only in GC

$$\omega = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle}$$

Hence scaled variance is not intensive, since different in different ensembles, but pseudo intensive



F.Becattini, L.Ferroni, T.Gabbriellini, Phys.Rev. C 72 (2005) 064904