Multiplicity Distributions in Statistical Ensembles

Michael Hauer

1Helmholtz Research School, University of Frankfurt, Frankfurt, Germany

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H-QM Helmholtz Research School
Quark Matter Studies
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5 Conclusion
Enhanced fluctuations are expected near a critical point of strongly interacting matter, or as a result of the onset of deconfinement.

Correlations, i.e. susceptibilities, are different in HRG and QGP.

Good experimental data are becoming available.

Calculate baseline (or statistical) fluctuations.

Can we understand properties of non-equilibrium models? (c.f. Volodya’s talk)

Test the statistical hadronization model.
### Statistical Ensembles

<table>
<thead>
<tr>
<th>GCE</th>
<th>CE</th>
<th>MCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z^{GCE}(V, T, \mu)$</td>
<td>$Z^{CE}(V, T, Q)$</td>
<td>$Z^{MCE}(V, E, Q)$</td>
</tr>
</tbody>
</table>

#### Grand Potential

$$
\Omega = -T \ln Z^{GCE}
$$

#### Helmholtz Free Energy

$$
F = -T \ln Z^{CE}
$$

#### Entropy

$$
S = \ln Z^{MCE}
$$

#### Multiplicity Distributions

$$
\mu_{CE} = \left( \frac{\partial F}{\partial Q} \right)_{V,T}
$$

#### Thermodynamic Limit

$$
\frac{E}{V} = \text{const, } \frac{Q}{V} = \text{const, } \Rightarrow T_{MCE} \rightarrow T, \quad \mu_{CE} \rightarrow \mu
$$

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**Michael Hauer**

Multiplicity Distributions in Statistical Ensembles

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The Boltzmann Massless Gas

Ultrarelativistic Gas of Neutral Particles

GCE Partition Function

\[ Z^{GCE}(V, T) = \exp \left[ V g \int \frac{d^3p}{(2\pi)^3} e^{-|p|/T} \right] = \exp \left[ V g \frac{T^3}{\pi^2} \right] \]

All Micro-states with N Particles

\[ Z^N(V, T) = \int_{-\pi}^{\pi} \frac{d\phi_N}{2\pi} e^{-iN\phi_N} \exp \left[ V g \frac{T^3}{\pi^2} e^{i\phi_N} \right] = \left( V g \frac{T^3}{\pi^2} \right)^N \]

GCE Multiplicity Distribution is Poissonian

\[ P_{GCE}(N) = \frac{Z^N(V, T)}{Z^{GCE}(V, T)} = \frac{(V g \frac{T^3}{\pi^2})^N}{N!} \exp \left( -V g \frac{T^3}{\pi^2} \right) \]
The Boltzmann Massless Gas

Joint Energy and Multiplicity Distribution

Ultrarelativistic Gas of Neutral Particles

All Micro-states with \( N \) Particles and Energy \( E \)

\[
Z_{N,E}^{V,T} = \left( \frac{gV}{\pi^2} \right)^N \frac{E^{3N-1}}{N!(3N-1)!} e^{-E/T} = \frac{Z_{MCE}^{V,E,N}}{Z_{MCE}^{V,E}} \exp \left[ Vg \int \frac{d^3p}{(2\pi)^3} e^{-|p|/T} e^{i|p|\phi_E} e^{i\phi_N} \right]
\]

All Micro-states with Energy \( E \)

\[
Z_{E}^{V,T} = \sum_{N=1}^{\infty} Z_{E,N}^{V,T} \quad \text{or} \quad Z_{MCE}^{V,E} = \sum_{N=1}^{\infty} Z_{MCE}^{V,E,N}
\]

MCE Multiplicity Distribution

\[
P_{MCE}(N) = \frac{Z_{N,E}^{V,E}}{Z_{E}^{V,E}} e^{+E/T} = \frac{Z_{MCE}^{V,E,N}}{Z_{MCE}^{V,E}}
\]

MCE \( P(N) \) can be expressed through conditional GCE \( P(N|E)! \)

M.H., V.V.Begun, M.I. Gorenstein, arXiv:0706.3290 [nucl-th]
Asymptotic Behavior

Mean Values

Equivalence of ensembles holds for mean values in the thermodynamic limit.

However .......

This seems not to apply to higher moments of a distribution!

Asymptotic Behavior

Express the Width by:

Scaled Variance

$$\omega = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle}$$

$$V, E \to \infty, \text{ and } E/V = \text{const}$$

$$\omega_{gce} = 1, \text{ but } \omega_{mce} \to 0.25$$

Hadron Resonance Gas Model

For instance:

Hadron Resonance Gas Model
- Used as an effective model of strong interaction
- Includes all hadrons and resonances up to $\sim 2$ GeV
- Depending on the version it assumes partial or complete chemical equilibrium
- However, thermal equilibrium is always assumed
Despite its simplicity, it fits a broad range of data with just 3 parameters. Interpretation, however, remains 'controversial'.

F. Becattini, private communication
Fourier Spectral Analysis of GCE Partition Function

\[ Z^{Q^j, E^k}(V, T, \mu_j) = \left[ \prod_{j=1}^{3} \int_{-\pi}^{\pi} \frac{d\phi_j}{2\pi} e^{-iQ^j\phi_j} \right] \left[ \prod_{k=1}^{4} \int_{-\infty}^{\infty} \frac{d\phi_k}{2\pi} e^{-iE^k\phi_k} \right] \exp \left[ V \sum_l \psi_l(\phi_j, \phi_k) \right] \]

Single Particle Partition Function

\[ \psi_l(\phi_j, \phi_k) = \frac{g_l}{(2\pi)^3} \int d^3p \ln \left( 1 \pm e^{-\frac{\sqrt{m_l^2+p^2}-\mu_l}{T}} e^{i(q^l_{j}\phi_j + q^l_{k}\phi_k)} \right)^{\pm 1} \]

It can be shown that generally:

\[ Z^{Q^j, E^k}(V, T, \mu_j) = Z^{MCE}(V, Q^j, E^k) e^{\frac{Q^j_{\mu_j}}{T}} e^{-\frac{E}{T}} \]

Some Definitions:

- \( Q^j = (B, S, Q) \)
- \( E^k = (E, P_x, P_y, P_z) \)
- \( q^l_{j} = (b_l, s_l, q_l) \)
- \( \epsilon^l_{j} = (\epsilon_l, p_x, p_y, p_z) \)

In the large volume limit \( Z^{Q^j, E^k}(V, T, \mu_j) \) converges to a Multivariate-Normal-Distribution. Finite Volume corrections are given in the form of Hermite polynomials of low order.

M.H., V.V.Begun, M.I. Gorenstein, arXiv:0706.3290 [nucl-th]
The arrows: indicate the effect of resonance decay

MCE
Energy conservation leads to correlation with (neutral) particles, thus suppresses final state fluctuations.

Scaled Variance Measured by NA49

**Thick Lines**

Acceptance Scaling

\[ \omega^{acc} = 1 - q + q\omega^{4\pi} \]

**Grey Area**

Guess-timate of effect of acceptance in momentum space in MCE


M.H., in preparation
Comparison to NA49 Fluctuation Data

Apart from Conservation Laws
Van der Waals Gas

GCE Van der Waals Gas

Has been used to:
model repulsive interactions between hadrons

- suppression of densities can be removed by rescaling the system volume
- suppression of fluctuations is qualitatively different
- could be a first step towards a simple model with a phase transition
- can be extended to include conservation laws

Boltzmann pion gas at $T = 160\, \text{MeV}$ and zero charge density.

- Each bin contains same fraction of total yield
- Bars indicate size of the bin

Energy and momentum conservation lead to suppressed multiplicity fluctuations at high $|y|$ and $p_T$.

M.H. in preparation
Apart from Conservation Laws

Phase Space Dependence

Momentum Cuts in UrQMD

UrQMD simulation of central Pb+Pb collision at b=0

Construction of bins is the same as before.


MCE suppression of fluctuations also in non-equilibrium systems?
Apart from Conservation Laws

Phase Space Dependence

Momentum Cuts in NA49 Data

UrQMD vs. NA49 158AGeV Pb+Pb data

Rapidity and transverse momentum dependence also seen in data!

MCE effects are of similar magnitude as proposed enhancement due to a phase transition / critical point!

B. Lungwitz, talk given at Workshop on Critical Point and Onset Deconfinement and private communication
Particle number fluctuations were discussed in different statistical ensembles and compared to NA49 data on nucleus-nucleus collisions.

Fluctuations are different in different ensembles!

GCE in clear contradiction to data!

Data as well as CE and MCE show suppressed fluctuations (with respect to a Poissonian).

Fluctuations are an important test for the statistical hadronization model.

We need to understand the role of experimental acceptance!
Acknowledgment

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for MANY discussions and his guidance!
What is the role of $T$ in MCE?

**Partition Function**

$$Z^E(V, T) \equiv Z^{MCE}(V, E) \ e^{-\frac{E}{T}}$$

**Entropy**

$$S = \ln \left( \frac{Z^E \ e^{\frac{E}{T}}}{Z^E} \right)$$

**Determine Equilibrium Temperature**

$$\left( \frac{\partial S}{\partial E} \right)_V = \frac{\frac{\partial Z^E}{\partial E} \ e^{\frac{E}{T}} + \frac{1}{T} Z^E \ e^{\frac{E}{T}}}{Z^E \ e^{\frac{E}{T}}} = \frac{1}{T_{MCE}}$$

$$T = T_{MCE} \implies \frac{\partial Z^E}{\partial E} = 0$$

$$\implies \text{Maximize GCE Partition Function for Energy } E$$

**GCE Partition Function**

$$Z^{GCE}(V, T) = 1 + \int_0^\infty dE \ Z^{MCE}(V, E) \ e^{-\frac{E}{T}} = 1 + \int_0^\infty dE \ Z^E(V, T)$$
What is the advantage of introducing T in MCE?

OR:

Why is it of advantage to define MCE multiplicity distributions through joint GCE distributions?

Principle Problem:

In CE and MCE calculations one has to deal with a heavily oscillating (or even irregular) integrand.

Our version is however very smooth!

Main contribution comes from small region around the origin.

Analytical expansion is possible

Numerical integration for large system becomes feasible!
Quality of Approximation

Large Volume Limit
- Multiplicity Distribution becomes Gaussian (CLT)
- $\mu \rightarrow \mu_{GCE}$
- $T \rightarrow T_{GCE}$

Finite Volume Correction
- Given by different order of Gram-Charlier expansion ($\text{GC3} - \text{GC5}$)
- In general only applicable to ‘body’ of distribution
- Very good description, even for small system size

M.H., V.V. Begun, M.I. Gorenstein, arXiv:0706.3290 [nucl-th]