Inhomogeneous chiral symmetry breaking phases

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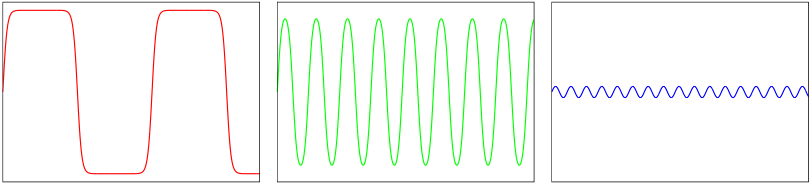

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H-QM Helmholtz Research School Quark Matter Studies

HGS-HIRe for FAIR Helmholtz Graduate School for Hadron and Ion Research

Motivation: the QCD phase diagram (so far ?)

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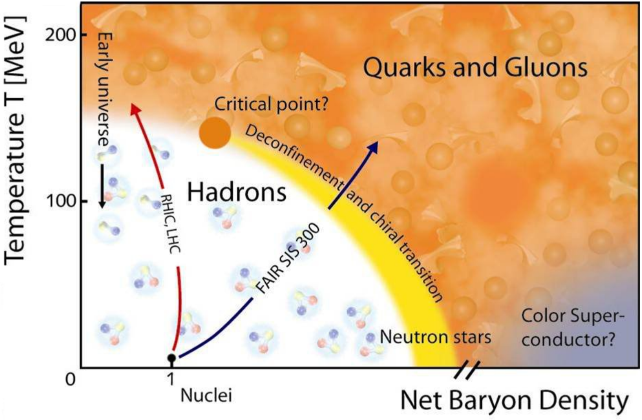

The model

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- Start from the usual $N_f = 2$ NJL Lagrangian

$$\mathcal{L}_{NJL} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi + G_s\left(\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}i\gamma^5\tau^a\psi\right)^2\right)$$

- Mean-field approximation
- Retain space dependence of the condensates

$$\langle\bar{\psi}\psi\rangle = S(\vec{x})\,, \qquad \langle\bar{\psi}i\gamma^5\tau^a\psi\rangle = P_a(\vec{x})$$

- Mean-field Lagrangian:

$$\mathcal{L}_{MF} = \bar{\psi}(x)\mathcal{S}^{-1}(x)\psi(x) - G_s\left(S(\vec{x})^2 + P(\vec{x})^2\right)$$

$$\mathcal{S}^{-1} = i\gamma^\mu\partial_\mu - m + 2G_s\left(S(\vec{x}) + i\gamma^5\tau^a P_a(\vec{x})\right) \equiv \gamma^0(i\partial_0 - \mathcal{H}_{MF})$$

Thermodynamic potential

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$$\Omega(T,\mu;S(\vec{x}),P(\vec{x})) = -\frac{T}{V}\mathrm{Log}\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\,\exp\left(\int_{x\in[0,\frac{1}{T}]\times V}(\mathcal{L}_{MF}+\mu\bar{\psi}\gamma^0\psi)\right)$$

$$= -\frac{TN_c}{V}\sum_n \mathrm{Tr}_{D,f,V}\,\mathrm{Log}\left(\frac{1}{T}(i\omega_n+\mathcal{H}_{MF}-\mu)\right)+\frac{G_s}{V}\int_V\left(S(\vec{x})^2+P(\vec{x})^2\right)$$

- If we can calculate the eigenvalues $\{E_n\}$ of $\mathcal{H}_{MF}$, it's

$$\Omega(T,\mu;M(\vec{x})) = -\frac{TN_fN_c}{V}\sum_{E_n}\mathrm{Log}\left(2\cosh\left(\frac{E_n-\mu}{2T}\right)\right)+\frac{1}{V}\int_V\frac{|M(\vec{x})-m|^2}{4G_s}$$

Having defined $M(\vec{x}) = m - 2G_S\left(S(\vec{x})+iP(\vec{x})\right)$

Eigenvalue problem

- Finding the eigenvalues of

$$\mathcal{H}_{MF} = -i\gamma^0\gamma^i\partial_i + \gamma^0 \left(m - 2G_s S(\vec{x}) - 2iG_s\gamma^5\tau^a P_a(\vec{x})\right)$$

- and minimizing the thermodynamic potential for an arbitrary $M(\vec{x})$

Eigenvalue problem

- Finding the eigenvalues of

$$\mathcal{H}_{MF} = -i\gamma^0\gamma^i\partial_i + \gamma^0\left(m - 2G_s S(\vec{x}) - 2iG_s\gamma^5\tau^a P_a(\vec{x})\right)$$

- and minimizing the thermodynamic potential for an arbitrary $M(\vec{x})$

is a very hard job!

- Try to simplify the problem:
- Consider lower-dimensional modulations

One-dimensional modulations:
$M(\vec{x}) \to M(z)$

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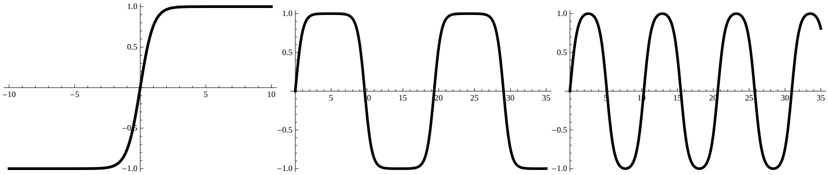
- Self-consistent real solutions known from studies of 1+1D Gross-Neveu model
  (M.Thies et al., Annals Phys. 314 (2004) 425-447, arXiv:hep-th/0402014)

$$M(z) = \Delta\sqrt{\nu}\,\mathrm{sn}(\Delta z|\nu)$$



- Analytical expression for the eigenvalue spectrum of $\mathcal{H}_{MF}\,[M(z)]$
- Minimization of $\Omega[M(z)]$ w.r.t. two parameters (chiral limit): $\Omega(\Delta,\nu)$

Results: NJL (chiral limit)

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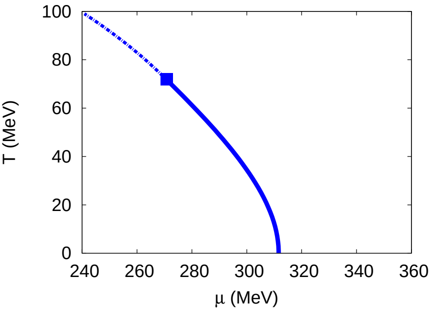

- Homogeneous only:

- First order phase transition
- ending at a critical point

Results: NJL (chiral limit)

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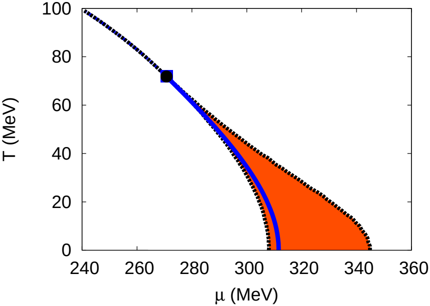

- Allow for inhomogeneous condensates:
- First order transition line covered by inhomogeneous phase
- All phase transitions are 2nd order
- Critical point $\rightarrow$ Lifschitz point

(D. Nickel, Phys.Rev.D80 074025, 2009 - arXiv:0906.5295)

What about pseudoscalar condensates?

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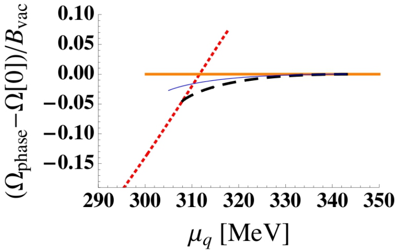
- Real modulations $\rightarrow P(x) = 0$

- Solitons: $M(z) \sim \Delta\sqrt{\nu}sn(z|\nu)$
- Chiral density wave: $M(z) = \Delta e^{iqz}$
- Homogeneous broken: $M(z) = \Delta$



(D. Nickel, PRD 80)

- (Real) solitons are always favored over chiral density wave!

Mass and density profiles ($T = m = 0$)

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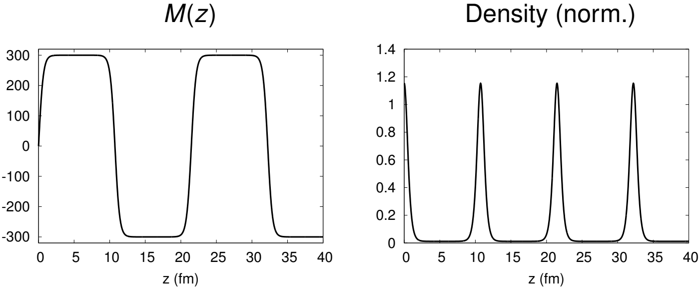
$$M(z) = \Delta\sqrt{\nu}\,\mathrm{sn}(\Delta z|\nu) = \begin{cases} \Delta\tanh(\Delta z) & \text{for } \nu \to 1 \\ \Delta\sqrt{\nu}\sin(\Delta z) & \text{for } \nu \to 0 \end{cases}$$



μ = 307.5 MeV

Mass and density profiles ($T = m = 0$)

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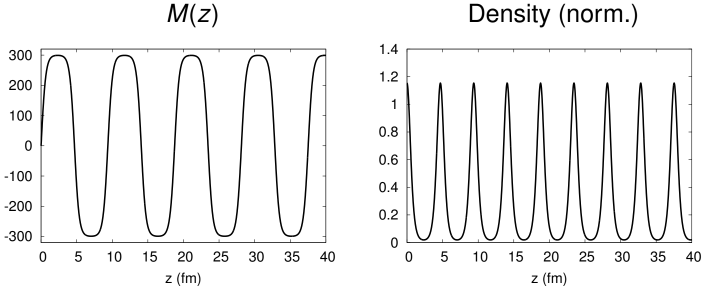

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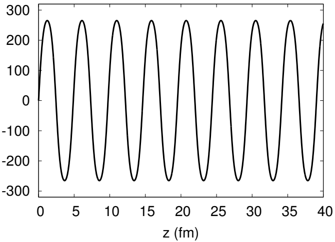
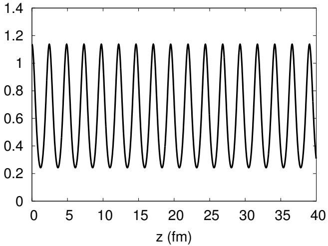
$M(z)$



Density (norm.)



μ = 309 MeV

Mass and density profiles ($T = m = 0$)

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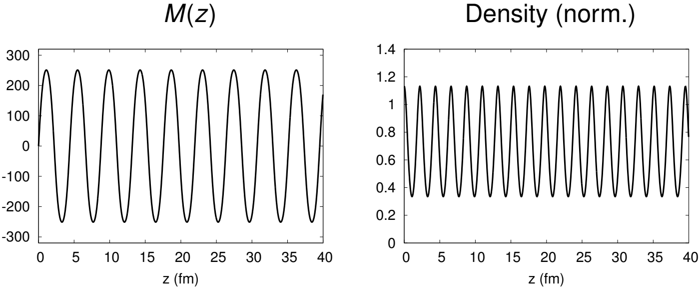
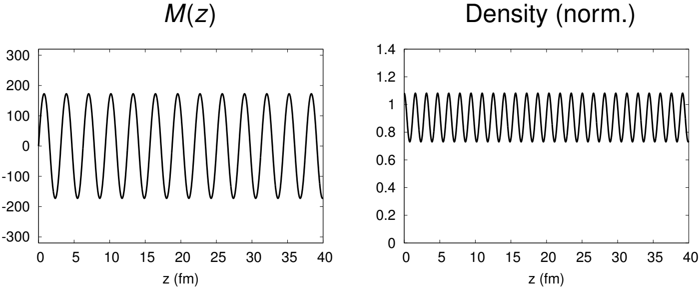


$\mu = 310$ MeV

Mass and density profiles ($T = m = 0$)

$$M(z) = \Delta\sqrt{\nu}\,\mathrm{sn}(\Delta z|\nu) = \begin{cases} \Delta\tanh(\Delta z) & \text{for } \nu \to 1 \\ \Delta\sqrt{\nu}\sin(\Delta z) & \text{for } \nu \to 0 \end{cases}$$

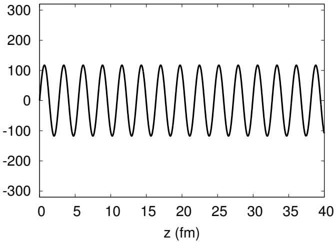


$\mu = 320$ MeV

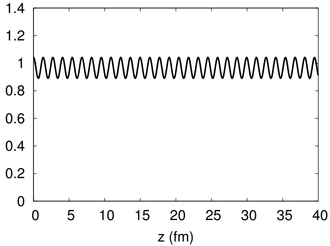
Mass and density profiles ($T = m = 0$)

$$M(z) = \Delta\sqrt{\nu}\,\mathrm{sn}(\Delta z|\nu) = \begin{cases} \Delta \tanh(\Delta z) & \text{for } \nu \to 1 \\ \Delta\sqrt{\nu}\sin(\Delta z) & \text{for } \nu \to 0 \end{cases}$$

$M(z)$



Density (norm.)



μ = 330 MeV

Mass and density profiles ($T = m = 0$)

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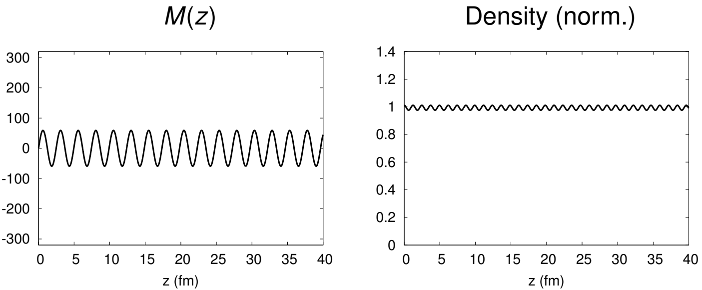
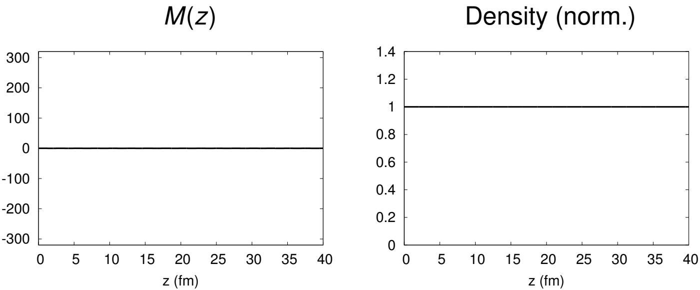
$$M(z) = \Delta\sqrt{\nu}\,\mathrm{sn}(\Delta z|\nu) = \begin{cases} \Delta\tanh(\Delta z) & \text{for } \nu \to 1 \\ \Delta\sqrt{\nu}\sin(\Delta z) & \text{for } \nu \to 0 \end{cases}$$



μ = 340 MeV

Mass and density profiles ($T = m = 0$)

$$M(z) = \Delta\sqrt{\nu}\,\mathrm{sn}(\Delta z|\nu) = \begin{cases} \Delta\tanh(\Delta z) & \text{for } \nu \to 1 \\ \Delta\sqrt{\nu}\sin(\Delta z) & \text{for } \nu \to 0 \end{cases}$$



μ = 345 MeV

Vercor interactions

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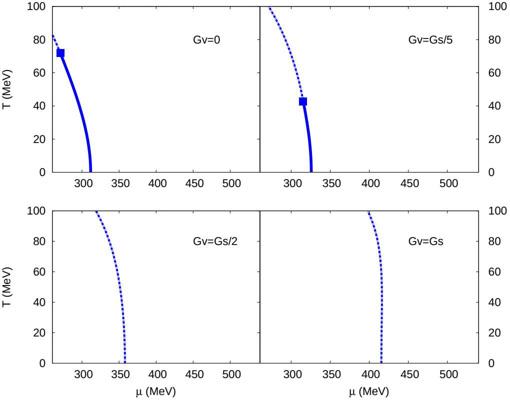
- Additional vector term: $\mathcal{L} = \mathcal{L}_{NJL} - G_V(\bar{\psi}\gamma^\mu\psi)^2$
- New mean field: $\bar{\psi}\gamma^\mu\psi \to \langle\bar{\psi}\gamma^\mu\psi\rangle \equiv n(\vec{x})\delta^{\mu 0}$ (density!)
- Introduce shifted chemical potential $\tilde{\mu}(\vec{x}) = \mu - 2G_V n(\vec{x})$
- Determine $\tilde{\mu}$ via $\frac{\delta\Omega}{\delta\tilde{\mu}} = 0$
- Sacrifice complete self-consistency: pick $\tilde{\mu} \equiv \langle\tilde{\mu}\rangle_z$ instead of $\tilde{\mu}(z)$
  - Most questionable in the inhomogeneous phase at low μ and T
  - More reliable close to the restored phase and the Lifshitz point

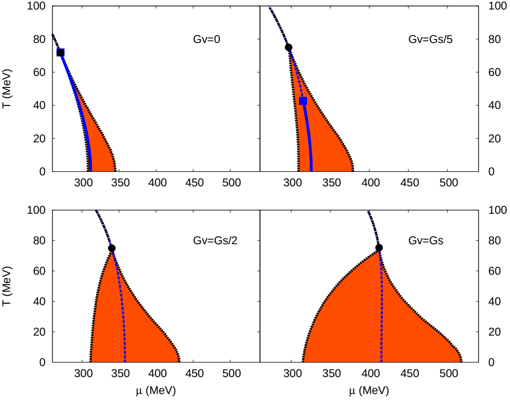
$$\Omega(T,\mu) \to \Omega(T,\tilde{\mu}) - \frac{(\mu-\tilde{\mu})^2}{4G_V}$$

Results: Vector interactions (Chiral limit)



- Homogeneous:
- Shift towards higher μ
- Strong G_V-dependence of the critical point

Results: Vector interactions (Chiral limit)



- Inhomogeneous:
- Stretch towards higher μ
- Lifshitz point at constant T
- Lifshitz and critical points split

Results: Vector interactions (Chiral limit)

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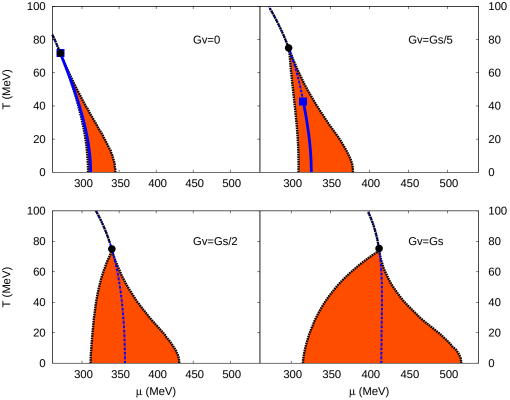
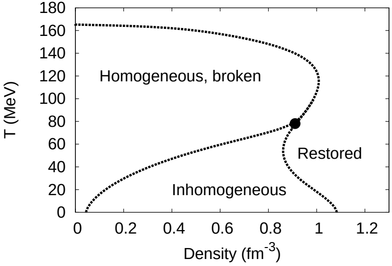


T vs $\langle n \rangle$



G_V-independent!

Chiral spiral

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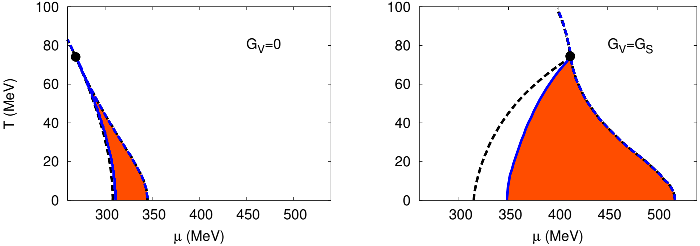
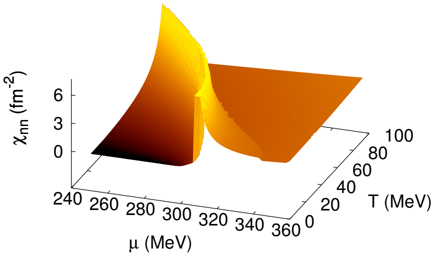
- How good is our $\tilde{\mu}(z) \rightarrow \langle\tilde{\mu}(z)\rangle$ approximation ?
- Cross-check: Chiral spiral $\rightarrow$ $M(z) = \Delta e^{iqz}$ $\rightarrow$ $n(z)$ = const.



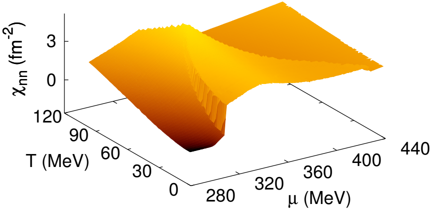
- Same qualitative behaviour as the solitonic solutions
- Lifshitz point at the same position
- Different (1st order) homogeneous $\rightarrow$ inhomogeneous transition line

Quark number susceptibilities

$$\chi_{nn} = -\frac{\partial^2\Omega}{\partial\mu^2} = \frac{\partial\bar{n}}{\partial\mu}$$



$G_V = 0$



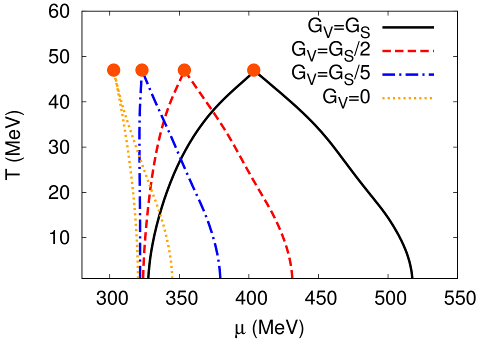
$G_V = G_S/2$

More phase diagrams: massive quarks

- Self-consistent solutions take the form

$$M(z) = \Delta \left(\sqrt{\nu}\,\mathrm{sn}(b|\nu)\mathrm{sn}(\Delta z|\nu)\mathrm{sn}(\Delta z + b|\nu) + \frac{\mathrm{cn}(b|\nu)\mathrm{dn}(b|\nu)}{\mathrm{sn}(b|\nu)} \right)$$

- Additional parameter: *b*
- Same qualitative features as $m = 0$
- Results for $m = 5$ MeV



Coupling with Polyakov Loop

- PNJL model:

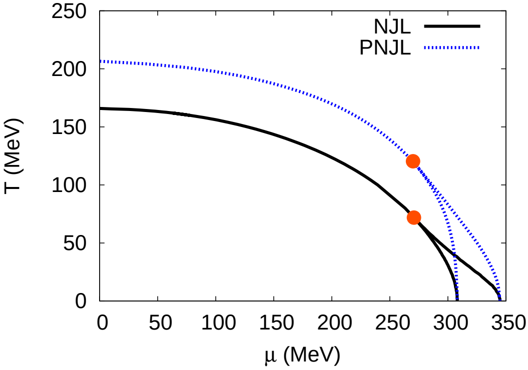
$$\mathcal{L}_{PNJL} = \bar{\psi}\left(i\gamma^{\mu}D_{\mu} - \hat{m}\right)\psi + G_s\left(\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}i\gamma^5\tau^a\psi\right)^2\right) - \mathcal{U}(L,\bar{L})$$

  - Covariant derivative: $D_{\mu} = \partial_{\mu} + iA_0\delta_{\mu 0}$
  - Polyakov loop: $L(\vec{x}) = \mathcal{P}exp[i\int_0^{1/T} d\tau A_4(\tau,\vec{x})]$, $\quad A_4(\tau,\vec{x}) = iA_0(t=-i\tau,\vec{x})$
  - Expectation values: $\ell = \frac{1}{N_c}\langle Tr_c L\rangle$, $\quad \bar{\ell} = \frac{1}{N_c}\langle Tr_c L^{\dagger}\rangle$
- Assumption: ℓ, $\bar{\ell}$ space-time independent
- Main effect:

$$N_c T\log\left(1 + e^{-\frac{E-\mu}{T}}\right) \rightarrow T\log\left(1 + e^{-3(E-\mu)/T} + 3\,\ell\, e^{-(E-\mu)/T} + 3\,\bar{\ell}\, e^{-2(E-\mu)/T}\right)$$

- Thermally excited quarks are suppressed at small ℓ, $\bar{\ell}$

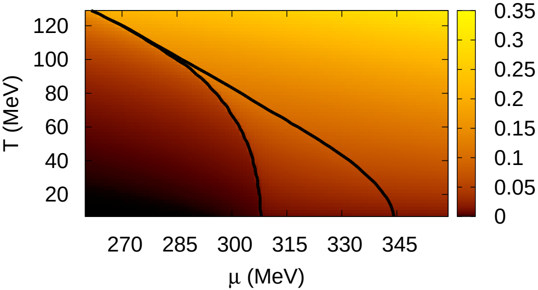
Results: PNJL (Chiral limit)



- Suppression of thermal effects
- Phase diagram stretched in T
- No qualitative change

Polyakov loop expectation value

- How good is our approximation of constant $\ell, \bar{\ell}$?



- Inhomogeneous regime: $\ell, \bar{\ell} \leq 0.2$
- Effects of neglecting spatial variations of $\ell, \bar{\ell}$ presumably small

1D Modulations:
What have we learned?

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- Self-consistent 1D spatial modulations can be studied by relying on analytical results from the study of the Gross-Neveu model
- Inhomogeneous 1D phases are favored over homogeneous ones in a region of the NJL phase diagram
- Extensions of the model (vector interactions, Polyakov loop) enhance the size of the inhomogeneous region
- The phase diagram is qualitatively altered !
- Only 2nd order phase transitions remain
- Vector interactions: $G_V > 0 \rightarrow$ CP disappears!

1D Modulations:
What next?

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- Relax our approximations:
  - Inhomogeneous $\tilde{\mu}$
  - Inhomogeneous Polyakov loop
- Three flavors
- Diquark pairing
- External magnetic field
- Two colors
- .....

Higher dimensional modulations ?

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- No analytical results to help us this time
- Brute force diagonalization of

$$\mathcal{H} = \gamma^0 \left[i\vec{\gamma} \cdot \vec{\partial} + m - 2G(S + i\gamma^5 P) \right]$$

- Expand $M(\vec{x})$ in a Fourier series. In momentum space:

$$\mathcal{H}_{p_{in},p_n} = \begin{pmatrix} -\vec{\sigma} \cdot \vec{p_{in}} \delta_{p_{in},p_{out}} & \sum_{\vec{q}} M_q \delta_{p_{out},p_{in}+q} \\ \sum_{\vec{q}} M_q \delta_{p_{out},p_{in}-q} & \vec{\sigma} \cdot \vec{p_{in}} \delta_{p_{in},p_{out}} \end{pmatrix}$$

- The inhomogeneous condensate couples different momenta

2D modulations

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- Assume spatial periodicity of the order parameter $M(x, y)$
- Grab your favorite Solid State Physics book
- Start exploiting the lattice symmetries:
  - $M(x, y)$ couples only momenta that differ by reciprocal lattice vectors
  - Project onto the Brillouin zone

$$\mathcal{H}_{\vec{p_{in}},\vec{p_{out}}} = \sum_{\vec{k}\in BZ} \mathcal{H}_{\vec{q_{in}},\vec{q_{out}}}(\vec{k}) \quad , \ \vec{q_{in}}, \vec{q_{out}} \in RL$$

  - $\mathcal{H}(k)$ becomes block-diagonal
- Diagonalize $\mathcal{H}$ numerically, minimize Ω

We want more!

Lower-dimensional modulations

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- Restrict to lower-dimensional spatial modulations
- $[H, P_\perp] = 0 \rightarrow$ Full spectrum from lower-dimensional eigenvalues λ
- Boost along the transverse directions:

$$(\lambda, \mathbf{0}) \rightarrow (\sqrt{p_\perp^2 + \lambda^2}, \mathbf{p}_\perp)$$

$$\Omega(T, \mu; M(\vec{x})) = -\frac{2TN_c}{V_{||}} \sum_\lambda \int \frac{d\vec{p}_\perp}{(2\pi)^{d_\perp}} \ln\left(2\cosh\left(\frac{\lambda\sqrt{1+\vec{p}_\perp^2/\lambda^2} - \mu}{2T}\right)\right) + \frac{1}{V}\int_V \frac{|M(\vec{x}) - m|^2}{4G_s} + \text{const.}$$

(D. Nickel, Phys.Rev.D80 074025, 2009 - arXiv:0906.5295)

1D modulations

- Restrict to one-dimensional modulations

$$M(\mathbf{x}) \to M(z)$$

- The Hamiltonian becomes

$$\tilde{H}_{MF} \to H'_{MF;1D} = \begin{pmatrix} H_{1D}(M(z)) & \\ & H_{1D}(M(z)^*) \end{pmatrix}$$

$$H_{1D}(M(z)) = \begin{pmatrix} -i\partial_z & M(z) \\ M(z)^* & i\partial_z \end{pmatrix} \quad \text{Gross-Neveu Hamiltonian}$$

Gross-Neveu

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- One of the simplest interacting fermionic field theories
- Defined in 1+1 dimensions

$$\mathcal{L}_{GN} = \bar{\psi} i \gamma^{\mu} \partial_{\mu} \psi + \frac{1}{2} g^2 (\bar{\psi}\psi)^2$$

- Hartree-Fock $\rightarrow$ recover SUSY QM-like equation
- Real self-consistent solutions of the form

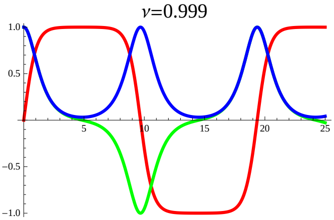
$$M(z) = \Delta \left(\nu \, \mathrm{sn}(b|\nu)\mathrm{sn}(\Delta z|\nu)\mathrm{sn}(\Delta z + b|\nu) + \frac{\mathrm{cn}(b|\nu)\mathrm{dn}(b|\nu)}{\mathrm{sn}(b|\nu)} \right)$$

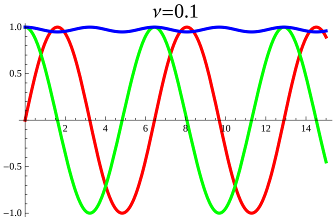
- Eigenvalue spectrum well known

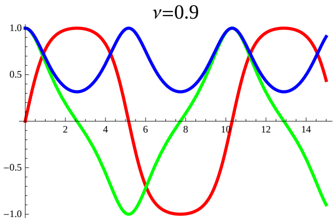
(M.Thies et al., Annals Phys. 314 (2004) 425-447, arXiv:hep-th/0402014)

Elliptic functions: sn$(z|\nu)$, cn$(z|\nu)$, dn$(z|\nu)$

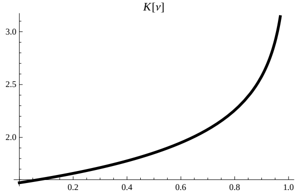
$\nu \in [0, 1]$







Period $\propto$ **K**(ν):



Thermodynamic potential

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$$\Omega_{MF}^{NJL}(T,\mu;\Delta,\nu,\delta) = -2N_c \int_0^\infty dE\, \tilde{\rho}(E;\nu,\Delta)\tilde{f}_{\text{bare}}\left(\sqrt{E^2+\delta\Delta^2}\right) + \frac{1}{4G_sP}\int_0^P dz\,|M(z)-m|^2 + \text{C}$$

$$\begin{aligned}
\tilde{f}_{\text{bare}}(x) &= \tilde{f}_{\text{UV}}(x) + \tilde{f}_{\text{medium}}(x) \\
\tilde{f}_{\text{UV}}(x) &= x \\
\tilde{f}_{\text{medium}}(x) &= T\ln\left(1+\exp\left(-\frac{x-\mu}{T}\right)\right) + T\ln\left(1+\exp\left(-\frac{x+\mu}{T}\right)\right)
\end{aligned}$$

$$\tilde{f}_{\text{UV}}(x) \to \tilde{f}_{\text{PV}}(x) = \sum_{j=0}^{3} c_j\sqrt{x^2+j\Lambda^2} \qquad (c_0=1,\, c_1=-3,\, c_2=3,\, c_3=-1)$$

Quark number susceptibility

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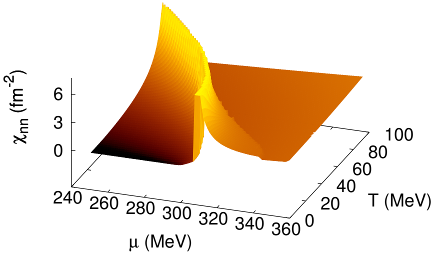
$$\chi_{nn} = -\frac{\partial^2 \Omega}{\partial \mu^2} = \frac{\partial n}{\partial \mu}$$



Susceptibilities