Heavy Probes in Heavy-Ion Collisions
Theory Part II

Hendrik van Hees

Justus-Liebig Universität Gießen

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1. Heavy-quark transport in the sQGP
   - Open heavy-flavor observables in heavy-ion collisions
   - Transport equations
   - The Fokker-Planck equation
   - Realization as Langevin process
   - Langevin simulation for heavy-ion collisions

2. In-medium interactions of heavy quarks I
   - Elastic pQCD heavy-quark scattering
   - Non-perturbative interactions: effective resonance model

3. Non-photonic electrons at RHIC
Heavy quarks in the sQGP

Hard production of HQs described by PDF’s + pQCD (PYTHIA)

$c, b$ quark

HQ rescattering in QGP: Langevin simulation
drag and diffusion coefficients from microscopic model for HQ interactions in the sQGP

Hadronization to $D, B$ mesons via quark coalescence + fragmentation

Semileptonic decay $\Rightarrow$ “non-photonic” electron observables
The relativistic Boltzmann equation

- describe heavy-quark scattering in the QGP by (semi-)classical transport equation
- \( f_Q(t, \vec{x}, \vec{p}) \): phase-space distribution of heavy quarks
- equation of motion for HQ-fluid cell at time \( t \) at \((\vec{p}, \vec{x})\):

  \[
  df_Q = dt \left( \frac{\partial}{\partial t} + \vec{v} \frac{\partial}{\partial \vec{x}} + \vec{F} \cdot \frac{\partial}{\partial \vec{p}} \right) f_Q
  \]

  - change of phase-space distribution with time (non-equilibrium)
  - drift of HQ-fluid cell with velocity \( \vec{v} = \vec{p}/E_{\vec{p}}, \quad E_{\vec{p}} = \sqrt{m_Q^2 + \vec{p}^2} \)
  - change of momentum with mean-field force, \( \vec{F} \)
  - change must be due to collisions with surrounding medium

  \[
  df_Q = C[f_Q] = \int d^3\vec{k} [w(\vec{p} + \vec{k}, \vec{k})f_Q(t, \vec{x}, \vec{p} + \vec{k}) - w(\vec{p}, \vec{k})f_Q(t, \vec{x}, \vec{p})]
  \]

  - \( w(\vec{p}, \vec{k}) \): transition rate for collision of a heavy quark with momentum, \( \vec{p} \) with a heat-bath particle with momentum transfer, \( \vec{k} \)
Transition rates

- relation to cross sections of microscopic scattering processes
- e.g., elastic scattering of heavy quark with light quarks

\[ w(\vec{p}, \vec{k}) = \gamma_q \int \frac{d^3 \vec{q}}{(2\pi)^3} f_q(\vec{q}) v_{\text{rel}}(\vec{p}, \vec{q} \rightarrow \vec{p} - \vec{k}, \vec{q} + \vec{k}) \frac{d\sigma}{d\Omega} \]

- \( \gamma_q = 2 \times 3 = 6 \): spin-color-degeneracy factor
- \( v_{\text{rel}} := \sqrt{(p \cdot q)^2 - (m_Q m_q)^2 / (E_Q E_q)} \); covariant relative velocity
- in terms of invariant matrix element

\[ C[f_Q] = \frac{1}{2E_Q} \int \frac{d^3 \vec{q}}{(2\pi)^3} \int \frac{d^3 \vec{p}'}{(2\pi)^3} \int \frac{d^3 \vec{q}'}{(2\pi)^3} \int \frac{d^3 \vec{p}}{(2\pi)^3} \int \frac{d^3 \vec{q}}{(2\pi)^3} \]

\[ \times \frac{1}{\gamma_Q} \sum_{c,s} |M(\vec{p}', \vec{q}') \leftrightarrow (\vec{p}, \vec{q})|^2 \]

\[ \times (2\pi)^4 \delta^{(4)}(p + q - p' - q')[f_Q(\vec{p}')f_q(\vec{q}') - f_Q(\vec{p})f_q(\vec{q})] \]

- \( \vec{p}, \vec{q} (\vec{p}', \vec{q}') \) initial (final) momenta of heavy and light quark
- momentum transfer: \( \vec{k} = \vec{q}' - \vec{q} = \vec{p} - \vec{p}' \)
- sum over all (“relevant”) scattering processes
The Fokker-Planck Equation

- **heavy quarks ↔ light quarks/gluons**: momentum transfers small
- \( w(\vec{p} + \vec{k}, \vec{k}) \): peaked around \( \vec{k} = 0 \)
- expansion of collision term around \( \vec{k} = 0 \)

\[
\begin{align*}
  w(\vec{p} + \vec{k}, \vec{k}) f_Q(\vec{p} + \vec{k}, \vec{k}) & \approx w(\vec{p}, \vec{k}) f_Q(\vec{p}) + \vec{k} \cdot \frac{\partial}{\partial \vec{p}} [w(\vec{p}, \vec{k}) f_Q(\vec{p})] \\
  & \quad + \frac{1}{2} k_i k_j \frac{\partial^2}{\partial \vec{p}_i \vec{p}_j} [w(\vec{p}, \vec{k}) f_Q(\vec{p})]
\end{align*}
\]

- collision term

\[
C[f_Q] = \int d^3\vec{k} \left[ k_i \frac{\partial}{\partial p_i} + \frac{1}{2} \frac{\partial^2}{\partial p_i \partial p_j} \right] [w(\vec{p}, \vec{k}) f_Q(\vec{p})].
\]
The Fokker-Planck Equation

- Boltzmann equation $\Rightarrow$ simplifies to Fokker-Planck equation

$$
\partial_t f_Q(t, \vec{x}, \vec{p}) + \frac{\vec{p}}{E_{\vec{p}}} \cdot \frac{\partial}{\partial \vec{x}} f_Q(t, \vec{x}, \vec{p}) = \frac{\partial}{\partial p_i} \left\{ A_i(\vec{p}) f_Q(t, \vec{x}, \vec{p}) \right. \\
+ \frac{\partial}{\partial p_j} [B_{ij}(\vec{p}) f_Q(t, \vec{p})] \left. \right\}
$$

- with drag and diffusion coefficients

$$
A_i(\vec{p}) = \int d^3 \vec{k} k_i w(\vec{p}, \vec{k}), \quad B_{ij}(\vec{p}) = \frac{1}{2} \int d^3 \vec{k} k_i k_j w(\vec{p}, \vec{k})
$$

- equilibrated light quarks and gluons: coefficients in heat-bath frame
- matter homogeneous and isotropic

$$
A_i(\vec{p}) = A(p) p_i, \quad B_{ij}(\vec{p}) = B_0(p) P_{ij}^\perp + B_1(p) P_{ij}^\parallel
$$

with

$$
P_{ij}^\perp(\vec{p}) = \frac{p_i p_j}{\vec{p}^2}, \quad P_{ij}^\parallel(\vec{p}) = \delta_{ij} - \frac{p_i p_j}{\vec{p}^2}
$$
Meaning of the Coefficients

- Simplified equation for momentum distribution, $F_Q(t, \vec{p})$
- Integrate Fokker-Planck equation over whole spatial volume:

$$F_Q(t, \vec{p}) = \int_V d^3 \vec{x} f_Q(t, \vec{x}, \vec{p}),$$

$$\int_V d^3 \vec{x} \text{div} \vec{x} \left[ \frac{\vec{p}}{E_{\vec{p}}} f(t, \vec{x}, \vec{p}) \right] = \int_{\partial V} d\vec{S} \cdot \left[ \frac{\vec{p}}{E_{\vec{p}}} f(t, \vec{x}, \vec{p}) \right] = 0 \Rightarrow$$

$$\frac{\partial}{\partial t} F_Q(t, \vec{p}) = \frac{\partial}{\partial p_i} \left\{ A_i(\vec{p}) F_Q(t, \vec{p}) + \frac{\partial}{\partial p_j} [B_{ij}(\vec{p}) F_Q(t, \vec{p})] \right\}$$

- most simple case in non-relativistic limit $A(\vec{p}) = A = \text{const}$, $B_0(\vec{p}) = B_1(\vec{p}) = B = \text{const}$

$$F_Q(t, \vec{p}) = \left\{ \frac{A}{2\pi D} \left[ 1 - \exp(-2\gamma t) \right] \right\}^{-3/2} \times \exp \left[ -\frac{A}{2B} \frac{[\vec{p} - \vec{p}_0 \exp(-At)]^2}{1 - \exp(-2\gamma t)} \right]$$
Meaning of the Coefficients

- solution: Gaussian with
  \[ \langle \vec{p}(t) \rangle = \vec{p}_0 \exp(-At), \quad \Delta \vec{p}^2(t) = \langle \vec{p}^2 \rangle - \langle \vec{p} \rangle^2 = \frac{3B}{A} [1 - \exp(-2At)]. \]

- \( A \): friction/drag coefficient \( \Rightarrow \) dissipation
- \( 1/A \): relaxation time to reach equilibrium
- \( B \): momentum-diffusion coefficient
- measures size of momentum fluctuations
- (result of random uncorrelated collisions of heavy quarks with medium)
- \( \Rightarrow \) effective description of collisions: white-noise-random force

- equilibrium limit \( (t \to \infty) \)

  \[ F_Q(t, \vec{p}) \approx \left( \frac{2\pi B}{A} \right)^{3/2} \exp \left( -\frac{A\vec{p}^2}{2B} \right) \]

- has to be Maxwell-Boltzmann distribution \( \Rightarrow \)

  \[ B = m_Q AT \]

- \( T \): given temperature of the QGP
- Einstein’s dissipation-flucutuation relation (1905)
Realization as Langevin process

- **Langevin process**: friction force + Gaussian random force
- in the (local) rest frame of the heat bath

\[
\begin{align*}
\frac{d\vec{x}}{dt} &= \frac{\vec{p}}{E_p}, \\
\frac{d\vec{p}}{dt} &= -A\vec{p}dt + \hat{C}\vec{w}\sqrt{dt}
\end{align*}
\]

- \(\vec{w}(t)\): Gaussian-distributed random variable

\[
\langle \vec{w}(t) \rangle = 0, \quad \langle w_j(t)w_k(t') \rangle = \delta(t - t')
\]

- \(\hat{C} = \hat{C}^t\): covariance matrix of random force
- stochastic process depends on choice of momentum argument of \(\hat{C}\)

\[
\hat{C} \rightarrow \hat{C}(t, \vec{x}, \vec{p} + \xi d\vec{p}), \quad \xi \in [0, 1]
\]

- usual values of \(\xi\)
  - \(\xi = 0\): pre-point Ito realization
  - \(\xi = 1/2\): Stratonovich realization
  - \(\xi = 1\): post-point Ito (Hänggi-Klimontovich) realization
Langevin ↔ Fokker-Planck

- **heavy-quark** phase-space distribution

\[ f_Q(t, \vec{x}, \vec{p}) = \left\langle \delta^{(3)}[\vec{x} - \vec{x}'(t)]\delta^{(3)}[\vec{p} - \vec{p}'(t)] \right\rangle \]  \hspace{1cm} (1)

- \([\vec{x}'(t), \vec{p}'(t)]\): trajectories according to **stochastic** Langevin process

\[ \begin{align*}
\text{d}\vec{x} &= \frac{\vec{p}}{E_p} \text{d}t, \\
\text{d}\vec{p} &= -A\vec{p} \text{d}t + \hat{C}\vec{w} \sqrt{\text{d}t}
\end{align*} \]  \hspace{1cm} (2)

- perform timestep of Eq. (1) using (2)

\[ \begin{align*}
\frac{\partial f_Q}{\partial t} + \frac{p_j}{E} \frac{\partial f_Q}{\partial x_j} &= \frac{\partial}{\partial p_j} \left[ \left( A p_j - \xi C_{lk} \frac{\partial C_{jk}}{\partial p_l} \right) f_Q \right] + \frac{1}{2} \frac{\partial^2}{\partial p_j \partial p_k} \left( C_{jl} C_{kl} f_Q \right) \\
\Rightarrow \quad C_{jk} &= \sqrt{2B_0} P_{jk}^\perp + \sqrt{2B_1} P_{jk}^\parallel
\end{align*} \]

- Form of Fokker-Planck equation ok, but how to chose \(\xi\)?
Langevin ↔ Fokker-Planck

- Choice of $\xi$: $f_Q \rightarrow$ Maxwell-Boltzmann distribution for $t \rightarrow \infty$:
  \[ f_{eq}^Q(p) \propto \exp(-\sqrt{\vec{p}^2 + m_Q^2}/T) \]

- Langevin process with $B_0 = B_1 = D(E) \Rightarrow C_{jk} = \sqrt{2D(E)}\delta_{jk}$
- MB distribution solution of stationary FP equation \(\Rightarrow\)
  \[ A(E)ET - D(E) + (1 - \xi)TD'(E) = 0 \]

- Simple choice: $\xi = 1$ (post-point Ito realization)
- Then simple Einstein dissipation-fluctuation relation
  \[ D = TEA \]

- For models for FP coefficients: relation not well satisfied for $B_1$
  \(\Rightarrow\) use $\xi = 1$ and $B_1 = TEA$
- Numerical check: Langevin simulation has right equilibrium limit
Langevin simulation for heavy-ion collisions

- need to simulate heavy-quark diffusion in sQGP
- “bulk” (light quarks + gluons) described by thermal fireball model
- flowing medium in local thermal equilibrium
- FP coefficients and Langevin process in restframe of the heat bath
- way out: boost to local heat-bath frame with flow velocity $v(t, \vec{x})$
- do time step to “update” momenta
- boost back to “lab frame”
- defines HQ distribution as “freezeout at constant lab time”
- NB: leads to covariant equilibrium distribution

$$dN_Q = \frac{\gamma Q}{(2\pi)^3}d^3\vec{x}(\text{hb}) \frac{d^3p}{p_0} p \cdot u(x) \exp\left(-\frac{p \cdot u(x)}{T(x)}\right)$$

- $u(t, \vec{x}) = [1, \vec{v}(t, \vec{x})]/\sqrt{1 - \vec{v}^2(t, \vec{x})}$: velocity-flow field (4-vector)
- $T(x)$: temperature field (4-scalar)
Fire-ball model

- **Elliptic fire-ball** parameterization fitted to hydrodynamical flow pattern [Kolb ’00]
  \[ V(t) = \pi(z_0 + v_z t)a(t)b(t), \quad a, b: \text{semi-axes of ellipse}, \]
  \[ v_{a,b} = v_\infty [1 - \exp(-\alpha t)] \pm \Delta v [1 - \exp(-\beta t)] \]

- **Isentropic expansion**: \( S = \text{const} \) (fixed from \( N_{ch} \))

- **QGP Equation of state**:
  \[ s = \frac{S}{V(t)} = \frac{4\pi^2}{90} T^3 (16 + 10.5 n_f^*), \quad n_f^* = 2.5 \]

- Obtain \( T(t) \Rightarrow A(t,p), B_0(t,p) \) and \( B_1 = TEA \)

- For semicentral collisions (\( b = 7 \text{ fm} \)): \( T_0 = 340 \text{ MeV} \), QGP lifetime \( \simeq 5 \text{ fm}/c \).

- Simulate FP equation as **relativistic Langevin process**
Initial conditions

- need initial $p_T$-spectra of charm and bottom quarks
- (modified) PYTHIA to describe exp. D meson spectra, assuming δ-function fragmentation
- exp. non-photonic single-$e^\pm$ spectra: Fix bottom/charm ratio

\[ \frac{1}{(2\pi p_T)} dN/dp_T \text{ [a.u.]} \]

\[ d+Au \ \sqrt{s_{NN}}=200 \text{ GeV} \]

\[ \sigma_{bb}/\sigma_{cc} = 4.9 \times 10^{-3} \]
Elastic pQCD processes

- Lowest-order matrix elements [Combridge 79]

- Debye-screening mass for $t$-channel gluon exch. $\mu_g = gT$, $\alpha_s = 0.4$

- not sufficient to understand RHIC data on “non-photonic” electrons [Moore, Teaney 2005]
Non-perturbative interactions: Resonance Scattering

- General idea: Survival of $D$- and $B$-meson like resonances above $T_c$
- model based on chiral symmetry (light quarks) HQ-effective theory
- elastic heavy-light-(anti-)quark scattering

$D, D', D_s$ resonances in sQGP

- parameters
  - $m_D = 2 \text{ GeV}, \Gamma_D = 0.4 \ldots 0.75 \text{ GeV}$
  - $m_B = 5 \text{ GeV}, \Gamma_B = 0.4 \ldots 0.75 \text{ GeV}$
total pQCD and resonance cross sections: comparable in size
BUT pQCD forward peaked ↔ resonance isotropic
resonance scattering more effective for friction and diffusion
Transport coefficients: pQCD vs. resonance scattering

- three-momentum dependence

- resonance contributions factor $\sim 2 \ldots 3$ higher than pQCD!
Transport coefficients: pQCD vs. resonance scattering

Temperature dependence

- resonances: $\Gamma = 0.4 \text{ GeV}$
- pQCD: $\alpha_s = 0.4$
- total

$D \ [\text{GeV/fm}]$

$\gamma \ [1/\text{fm}]$

T [GeV]

0.2 0.25 0.3 0.35 0.4
Spectra and elliptic flow for heavy quarks

\[ \mu_D = gT, \quad \alpha_s = g^2/(4\pi) = 0.4 \]

- resonances \( \Rightarrow \) \( c \)-quark thermalization without upscaling of cross sections
- Fireball parametrization consistent with hydro

\[ \mu_D = 1.5T \quad \text{fixed} \]

- spatial diff. coefficient:
  \[ D = D_s = \frac{T}{mA} \]
  \[ 2\pi T D \approx \frac{3}{2\alpha_s^2} \]
Spectra and elliptic flow for heavy quarks

Au-Au $\sqrt{s}=200$ GeV (b=7 fm)

LO QCD
[Moore, Teaney '04]

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Heavy Probes in HICs (Theory II)
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Observables: $p_T$-spectra ($R_{AA}$), $v_2$

- **Hadronization:** Coalescence with light quarks + fragmentation \[
\Leftrightarrow c\bar{c}, b\bar{b} \text{ conserved}
\]
- single electrons from decay of $D$- and $B$-mesons

Without further adjustments: data quite well described

[HvH, V. Greco, R. Rapp, Phys. Rev. C 73, 034913 (2006)]
Observables: $p_T$-spectra ($R_{AA}$), $v_2$

- Hadronization: Fragmentation only
- single electrons from decay of $D$- and $B$-mesons

- coalescence brings up both, $R_{AA}$ and $v_2$
- due to additional momentum kick from light quarks
Observables: $p_T$-spectra ($R_{AA}$), $v_2$

- Central Collisions
- Single electrons from decay of $D$- and $B$-mesons

Coalescence + Fragmentation

Fragmentation only

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Comparison to newer data

(a) 0−10% central

- Armesto et al. (I)
- van Hees et al. (II)

- \[ \frac{3}{2\pi T} \] Moore &
- \[ \frac{12}{2\pi T} \] Teaney (III)

Au+Au @ \( \sqrt{s_{NN}} = 200 \text{ GeV} \)

(b) minimum bias

- \( \pi^0 R_{AA}, p_T > 4 \text{ GeV/c} \)
- \( \pi^0 v_2, p_T > 2 \text{ GeV/c} \)
- \( e^\pm R_{AA}, e^\pm v_{2HF} \)

PHENIX Collaboration
PRL 98 172301 (2007)
Transport properties of the sQGP

- spatial diffusion coefficient: Fokker-Planck ⇒ \( D_s = \frac{T}{m_A} = \frac{T^2}{D} \)
- coupling strength in plasma: viscosity/entropy density, \( \frac{\eta}{s} \)

\[
\frac{\eta}{s} \simeq \frac{1}{2} T D_s \quad \text{(AdS/CFT)}, \quad \frac{\eta}{s} \simeq \frac{1}{5} T D_s \quad \text{(wQGP)}
\]

\[ \text{charm quarks} \]

\[ \text{T-mat + pQCD} \quad \text{reso + pQCD} \quad \text{pQCD} \quad \text{pQCD run. } \alpha_s \quad \text{KSS bound} \]

[\text{Lacey, Taranenko (2006)}]
Instead of a summary: Questions

- How relate (semi-)classical transport models the behavior of many-body systems to microscopic constituents?
- Why can for heavy quarks the transport equations be approximated by a Fokker-Planck equation?
- How are medium properties characterized within the Fokker-Planck equation?
- What is the microscopic picture arising from the Langevin equation?
- What can we learn within this theoretical picture from heavy-quark observables in heavy-ion collisions?
- Which properties of the sQGP can be extracted from that model?
Summary

- **Boltzmann Transport Equations**
  - can be derived from classical mechanics or quantum-many-body theory
  - (semi-)classical statistical description of interacting many-body systems
  - equations for single-particle phase-space distribution
  - collision term: transition probabilities from microscopic cross sections
  - many-body systems ⇔ microscopic properties of constituents

- **Fokker-Planck Equations**
  - heavy particles immersed in medium of light particles
  - momentum transfer in single collision small ⇒ integro-differential Boltzmann equation ⇒ partial differential equation
  - HQ-medium interactions ⇒ friction/drag coefficient + diffusion coefficients
  - related by Einstein dissipation-fluctuation relation
**Summary**

- **Langevin Equations**
  - stochastic differential equation equivalent to Fokker-Planck equation
  - drag/friction force + random forces = uncorrelated Gaussian noise
  - depends on realization of stochastic process
  - right process ⇒ equilibrium limit = relativistic MB distribution
  - application to flowing sQGP

- **Heavy-quark interactions in the sQGP I**
  - elastic scattering with light quarks and gluons: pQCD + screening
  - resonance scattering with light (anti-)quarks

- **Non-photonic single electron observables**
  - $R_{AA}(p_T)$ and $v_2(p_T)$ of electrons from $D$- and $B$-meson decays
  - Langevin simulation → coalescence+fragmentation hadronization → semi-leptonic decay
  - pQCD (with realistic $\alpha_s$) too weak
  - with resonance-scattering interactions good description of data