2.4 Scattering theory

given a Hamiltonian in \( F \) space,
\[
H(t) = \int d^3r \, \psi^* \left( \psi \right) \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - V \right) \psi \left( \psi \right)
\]

is no external forces and no explicit time dependence of interaction,

want to describe scattering event of two particles starting at time \( t_0 \) at \( r_1, r_2 \) and ending up at time \( t_2 \) at \( r_1', r_2' \)

\( \psi \) must be known probability amplitude
\[
\psi \left( r_1, r_1', r_2, r_2' \right) = \langle r_1', r_2' | \psi \rangle \langle r_1, r_2 | \psi \rangle
\]

In principle, solution via time evolution operator (cf. prop)

\[
\{ r_1, r_1', r_2, r_2' \} = U \left( r_1, r_1', r_2, r_2' \right)
\]

\( U \) (t, 0) evolution,
\[
-\partial^2_{t^2} U \left( t, 0 \right) = H \left( t \right) U \left( t, 0 \right)
\]

\( \Rightarrow \) \( U \left( t, 0 \right) = \exp \left( i H (t) t \right) \) satisfies

\[
-\partial^2_{t^2} U \left( t, 0 \right) = H \left( t \right) \exp \left( i H (t) t \right)
\]

for exact at least \( H \) no time dependent since

\[
\frac{d}{dt} H \left( t \right) = \frac{d}{dt} \left[ H \left( t \right), H \left( t \right) \right] = 0
\]

\( \Rightarrow \) \( H \left( t \right) = H \)

\( \Rightarrow \) \( U \left( t, 0 \right) = \exp \left( i H (t) t \right) \)
\[ p(F_z, F_{-z}, k_x, k_y, k_z) = \langle r_x, r_y, r_z \mid \mathcal{U}_z(t_z, t) \mid F_x, F_y, F_z \rangle \]
\[ = \langle 0 \mid \phi(r_x, r_y) \phi(r_z) \exp(-iH(t_x, t_y)/\hbar) \phi(\mathbf{r}_x, \mathbf{r}_y) \phi(r_z) \rangle \phi(\mathbf{r}_x, \mathbf{r}_y, r_z) \]

with
\[ H = H(t_z) = \int d^3x \phi^+(r_x, r_y) \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r_x^2} \right) \phi(r_x, r_y) \]
\[ \text{and} \]
\[ H_{\text{int}} = \frac{1}{2} \int d^3x d^3y \phi^+(r_x, r_y) \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r_x^2} \right) \phi^+(r_x, r_y) V_0(\mathbf{r}_x, \mathbf{r}_y) \phi(r_x, r_y) \phi(r_x, r_y) \phi(r_x, r_y) \]

now move annihilation operators \( \phi \) to the right and creation operators \( \phi^+ \) to the left, using the equal-time (anti-)commutation relations... \(^2\)

... But: \( \exp(-iH(t_x, t_y)/\hbar) \) contain infinitely many terms!

In practice (for small \( V_0 \)) use perturbation theory.

interaction picture

\[ H = H_0 + H_{\text{int}} \]

with
\[ H_0 = \int d^3x \phi^+(r_x, r_y) \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r_x^2} \right) \phi(r_x, r_y) \]

and
\[ H_{\text{int}} = \frac{1}{2} \int d^3x d^3y \phi^+(r_x, r_y) \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r_x^2} \right) \phi^+(r_x, r_y) V_0(\mathbf{r}_x, \mathbf{r}_y) \phi(r_x, r_y) \phi(r_x, r_y) \phi(r_x, r_y) \]

note: We have dropped the time dependence of the \( \phi \) and \( \phi^+ \)

here since it does not matter at which time they are evaluated - since \( H \) is time-independent. Below we will choose a convenient time to evaluate the \( \phi \) and \( \phi^+ \).
in an interaction.

When we continue with the general case, let us first study
in more detail the case of an interaction, i.e.

\[ V_x = 0 \Rightarrow H_{\text{ext}} = 0 \Rightarrow H + H_0 \]

we recall from page 123 (for \( V_x = 0 \))

\[ \left[ H_0, \Phi(\vec{r}, x) \right] = \frac{d^2}{dx^2} \Phi(\vec{r}, x) \]

introduce the Fourier transform

\[ a(\vec{p}, x) = \int d^3r e^{i\vec{p}\cdot\vec{r}} \Phi(\vec{r}, x) \]

\[ \Phi(\vec{p}, x) = \int d^3r e^{-i\vec{p}\cdot\vec{r}} a(\vec{p}, x) \]

\[ H_0 = \int d^3p \left( \frac{p^2}{2m} + \frac{e^2}{4\pi} a(\vec{p}, x) \right) \]

\[ \int d^3p \Phi(\vec{p}, x) = \int d^3p e^{i\vec{p}\cdot\vec{r}} \]

\[ \int d^3p a(\vec{p}, x) = \frac{e}{\sqrt{\pi}} \delta(\vec{p}) \]

\[ \int d^3p \Phi(\vec{p}, x) = \frac{e}{\sqrt{\pi}} \delta(\vec{p}) a(\vec{p}, x) \]

\[ \frac{\partial}{\partial t} a(\vec{p}, x) = \frac{1}{\hbar} \left[ H(\vec{p}), a(\vec{p}, x) \right] = \frac{e}{\hbar} E_{\vec{p}} a(\vec{p}, x) \]

\[ \frac{\partial}{\partial t} \Phi(\vec{p}, x) = e^{i\vec{p}\cdot\vec{r}} \frac{\partial}{\partial t} a(\vec{p}, x) \]

\[ \frac{\partial}{\partial t} \Phi(\vec{p}, x) = \frac{E_{\vec{p}}}{\hbar} e^{i\vec{p}\cdot\vec{r}} a(\vec{p}, x) \]

\[ \frac{\partial}{\partial t} \Phi(\vec{p}, x) = \frac{E_{\vec{p}}}{\hbar} e^{i\vec{p}\cdot\vec{r}} a(\vec{p}, x) \]

The equation of motion for \( a(\vec{p}, x) \) for case without interaction

(1) page 123

\[ \frac{\partial}{\partial t} a(\vec{p}, x) = \frac{1}{\hbar} \left[ H_0, a(\vec{p}, x) \right] = \frac{e}{\hbar} E_{\vec{p}} a(\vec{p}, x) \]

\[ \frac{\partial}{\partial t} \Phi(\vec{p}, x) = e^{i\vec{p}\cdot\vec{r}} \frac{\partial}{\partial t} a(\vec{p}, x) \]

\[ \frac{\partial}{\partial t} \Phi(\vec{p}, x) = \frac{E_{\vec{p}}}{\hbar} e^{i\vec{p}\cdot\vec{r}} a(\vec{p}, x) \]

\[ \frac{\partial}{\partial t} \Phi(\vec{p}, x) = \frac{E_{\vec{p}}}{\hbar} e^{i\vec{p}\cdot\vec{r}} a(\vec{p}, x) \]

\[ \frac{\partial}{\partial t} \Phi(\vec{p}, x) = \frac{E_{\vec{p}}}{\hbar} e^{i\vec{p}\cdot\vec{r}} a(\vec{p}, x) \]

\[ \frac{\partial}{\partial t} \Phi(\vec{p}, x) = \frac{E_{\vec{p}}}{\hbar} e^{i\vec{p}\cdot\vec{r}} a(\vec{p}, x) \]
With the time translation operator
\[ U_0(t, t') = \exp \left( i \frac{H_0(t, t')}{\hbar} \right) \]
we can also write (cf. page 23)
\[ \tilde{\psi}(\vec{x}, t_2) = U_0(t_2, t_1) \tilde{\psi}(\vec{x}, t_1) U_0^\dagger(t_1, t_1) \]
and
\[ a(\vec{p}, t_2) = U_0(t_2, t_1) a(\vec{p}, t_1) U_0^\dagger(t_2, t_1) \]
\[ e^{i \frac{E_0(t_1, t_1)}{\hbar} \vec{p} \cdot \vec{A}(t_1)} a(\vec{p}, t_1) \]

\(=\) in this case, i.e. without interaction, we can evaluate
interaction. Let
\[ e^{i \frac{H(t_1, t_1)}{\hbar} \vec{A}(t_1)} a(\vec{p}, t_1) e^{-i \frac{H(t_1, t_1)}{\hbar} \vec{A}(t_1)} \]

in spite of the fact that there are infinitely many operators
in \( e^{i \frac{H(t_1, t_1)}{\hbar} \vec{A}(t_1)} \)
so much more complicated for interaction case.
We now come back to the case with interaction, but with \( V_2 \) such that perturbation theory can give reasonable answers.

We can expect that
\[
p(\xi, t_2) = U(t_2, t_1) U^*(t_1, t_0) p(\xi, t_0) U(t_2, t_1)
\]
is close to (but not equal to) the limit
\[
U_0(t_2, t_1) p(\xi, t_0) \overline{U_0^*(t_1, t_0)}
\]
(recall \( U(t, t') = \exp \left(i \int_{t'}^{t} H(x') dx' \right) \))
\[
\text{and } U_0(t, t') = \exp \left(i \int_{t'}^{t} H_0(x') dx' \right)
\]

To introduce \( \mathbb{P}_D \left( \xi, t \right) = U_0(t, t_0) p(\xi, t_0) U_0^*(t, t_0) \)
\[
\Rightarrow \mathbb{P}(\xi, t) = U(t, t_0) p(\xi, t_0) U(t_0, t)
\]
\[
= U(t, t_0) U_0^*(t, t_0) \mathbb{P}_D \left( \xi, t \right) U_0(t, t_0) U(t_0, t)
\]
\[
= \mathbb{P}_{D_0}(t, t_0)
\]

Note: \( \mathbb{P}_{D_0}(t, t_0) \) can be calculated exactly from \( p(\xi, t_0) \) via the Fourier decomposition (ref. pages 23, 24):
\[
\overline{a}(\xi, t_0) = \int dx^* a^{*}\xi x^* p(\xi, x^*)
\]
\[
\Rightarrow \mathbb{P}_{D_0}(\xi, t_0) = \int \frac{d^3 k}{(2\pi)^3} e^{-i k \cdot \xi} \overline{q}(\xi, t_0)
\]

if \( \overline{a} \) is evaluated at the same time on \( a^* \), i.e. at \( t = t_0 \),
\[
\Rightarrow \mathbb{P}(\xi_1, \xi_2, t_0, \xi_3, \xi_4, t_1) = \langle 0 | p(\xi_3, t_2) p(\xi_2, t_1) p(\xi_1, t_0) p(\xi_4, t_1) | 0 \rangle
\]
\[
= \langle 0 | \overline{U}_D(t_0, t_1) p_0(t_1, t_2) p_0(t_2, t_3) \overline{U}_0(t_3, t_1) p_0(t_1, t_2) p_0(t_2, t_3) | 0 \rangle
\]
\[
\Rightarrow \mathbb{P}_{D_0}(\xi_1, t_0) = \mathbb{P}(\xi_1, t_0)
The operator $\hat{U}_D$ satisfies

\[
\frac{\partial}{\partial t} \hat{U}_D(t, t) = i \hat{H}_D(t, t) \hat{U}_D(t, t)
\]

\[
= -e^{iH_0(t, t)\frac{\tau_0}{\hbar}} \hat{H}_D e^{-iH_0(t, t)\frac{\tau_0}{\hbar}} + e^{i\hat{H}_0(t, t)\tau_0} \hat{H}_D e^{-i\hat{H}_0(t, t)\tau_0}
\]

\[
= \hat{U}_0(t, t) \hat{H}_D \hat{U}_0^+(t, t) \hat{U}_D(t, t)
\]

\[
= \hat{U}_0(t, t) \hat{U}_0^+(t, t) \hat{U}_D(t, t)
\]

Note:

1. In time independent, $\hat{U}_0$ and $\hat{U}_0^+$ are not separately time independent.

2. To calculate $\hat{U}_D(t, t)$ use the non-interacting case and then to take $\hat{U}_0$ at time $t_1$ and $\hat{U}_0^+$ at time $t_1$.

3. For a specific $\hat{H}_D(t, t)$

4. The operator $\hat{H}_D(t, t)$ is given by

\[
\hat{H}_D(\hat{\tau}, \hat{\tau}) = \frac{\hat{A}^+ \hat{A}}{\tau} + \sum_{t_1, t_2} \hat{A}^+(t_1, \tau) \hat{A}(t_2, \tau) + \hat{A}(t_1, \tau) \hat{A}^+(t_2, \tau)
\]

5. For a specific $\hat{H}_D(t, t)$

\[
\hat{U}_D(t, t) = T \exp \left(-\frac{i}{\hbar} \int_0^\tau d\tau \hat{H}_D(\tau) \hat{U}_0(\tau, \tau) \hat{U}_0^+(\tau, \tau)
\]

\[
= \hat{U}_0(t, t) \hat{U}_0^+(t, t) \hat{U}_D(t, t)
\]
\[
\text{since } H_D(0) = 0
\]
\[
\Rightarrow \langle 0 | \hat{u}_D^+(\mathbf{x}, t_0) | 0 \rangle = 0
\]
\[
\Rightarrow \langle 0 | \hat{u}_D^+(\mathbf{x}, t_0) | 0 \rangle = 0
\]
\[
\Rightarrow p(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_3', \mathbf{r}_4', t_0) = 0,
\]
\[
\Rightarrow 0 = \langle 0 | p_D(\mathbf{r}_1', t_0) p_D(\mathbf{r}_2', t_0) \rangle \text{ Temp } \left( \sum_{\mathbf{b}} \langle \mathbf{b} | H_D(\mathbf{b}) | \mathbf{b} \rangle \right) p_D(\mathbf{r}_1', t_0) p_D(\mathbf{r}_2', t_0) | 0 \rangle
\]
This expression should be compared to the one at page 85, top.

There we had interaction operator \( p, \phi_D^+ \) and the full Hamiltonian. Here we have an interacting operator \( p_D, \phi_D^+ \) and the interaction part \( H_D \) only.

Write down the first few terms explicitly:

(1) \text{First order in } V_D \text{ from } \text{Temp } \left( \sum_{\mathbf{b}} \langle \mathbf{b} | H_D(\mathbf{b}) | \mathbf{b} \rangle \right) \Rightarrow \text{All}

\[
\Rightarrow 0 = \langle 0 | p_D(\mathbf{r}_1', t_0) p_D(\mathbf{r}_2', t_0) p_D(p_D(\mathbf{r}_1', t_0) | 0 \rangle
\]

\text{Strategy for evaluation:}

- \text{Express } p_D(\mathbf{r}_1', t_0) \text{ in terms of } c(\mathbf{p}, t_0) \text{ (cf. page 88)}
- \text{Express } c(\mathbf{p}, t_0) \text{ in terms of } p_D(\mathbf{r}_1', t_0)
- \text{Insert } p_D(\mathbf{r}_1', t_0) \text{ with } p_D(\mathbf{r}_1', t_0)

\Rightarrow \text{not worked out here explicitly}
\( \text{2.} \) First order in \( V \): from linear term in \( T \), \( \exp(\ldots) \)

\[ \langle 0 | T \left( \prod_i f_i \left( \frac{x_i}{\hbar} \right) f_i^\dagger \left( \frac{x_i}{\hbar} \right) \right) | 10 \rangle \]

with \( H_0(t) \) given at page 282 bottom in terms of \( f_0, f_0^\dagger \).

Strategy for evaluation: same as done (6).

\( \text{3.} \) Second order in \( V \):

\[ \langle 0 | T \left( \prod_i f_i \left( \frac{x_i}{\hbar} \right) f_i^\dagger \left( \frac{x_i}{\hbar} \right) \prod_j \left( \frac{x_j}{\hbar} \right)^n \right) \exp \left( \int \frac{dt}{\hbar} H_0(t) \right) f_0 \left( \frac{x_1}{\hbar} \right) f_0^\dagger \left( \frac{x_1}{\hbar} \right) | 10 \rangle \]

Challenge: search time ordered operators such that creation operators are to the left and annihilation operators are to the right.

\( \Rightarrow \) White's theorem.

Note: under the cases 2 and 3, can be covered by the formalism which we present on the next page, since

\[ \langle 0 | T \left( \prod_i f_i \left( \frac{x_i}{\hbar} \right) f_i^\dagger \left( \frac{x_i}{\hbar} \right) \right) \exp \left( \int \frac{dt}{\hbar} H_0(t) \right) f_0 \left( \frac{x_1}{\hbar} \right) f_0^\dagger \left( \frac{x_1}{\hbar} \right) | 10 \rangle \]

with \( T(A(x), B(x)) = T(B(x), A(x)) A(x) B(x) - B(x) A(x) T(A(x), B(x)) \),

\( A, B \) brane/anti-brane creation or annihilation operators.
Weick's theorem:

At $0_1, 0_2, ..., 0_n$ free annihilation operators (one might be excited, some annihilation operators) in a non-interacting theory introduce "normal ordering":

The normal ordered product: $0_1^* 0_2^* ... 0_n^*$ in the expression obtained from $0_1 0_2 ... 0_n$ if all annihilation operators are moved to the right, for fermions one adds a minus sign for every required interchanging.

Examples:

\[ \phi(\vec{r}, t) \phi^*(\vec{r}', t) = \phi(\vec{r}, t) \phi^*(\vec{r}', t) \]
\[ \phi(\vec{r}, t) \phi^*(\vec{r}', t) = \phi^*(\vec{r}', t) \phi(\vec{r}, t) \]
\[ \phi^*(\vec{r}, t) \phi(\vec{r}', t) \phi^*(\vec{r}'', t) = \phi^*(\vec{r}', t) \phi(\vec{r}, t) \phi^*(\vec{r}'', t) \]
\[ H_{\text{int}} = \frac{1}{2} \int d\vec{r} d\vec{r}' V_\phi(\vec{r}, \vec{r}') \phi(\vec{r}, t) \phi(\vec{r}', t) \]

Weick's theorem:

\[ \Gamma(0_1, ..., 0_n) = : \phi_1 ... \phi_n : \]
\[ = \{ \Gamma(0_1, 0_2) \} \{ \Gamma(0_3, ..., 0_n) \} + \text{permutations} \]
\[ + \{ \Gamma(0_1, 0_2) \} \{ \Gamma(0_3, ..., 0_n) \} \{ \Gamma(0_1, 0_2) \} \{ \Gamma(0_3, ..., 0_n) \} + \text{permutations} \]
\[ + \{ \Gamma(0_1, 0_2, 0_3, 0_4) \} \{ \Gamma(0_5, ..., 0_n) \} + \text{permutations for } n \text{ even} \]
\[ + \{ \Gamma(0_1, 0_2) \} \{ \Gamma(0_3, ..., 0_n) \} \{ \Gamma(0_1, 0_2, 0_3, 0_4) \} \{ \Gamma(0_5, ..., 0_n) \} + \text{permutations for } n \text{ odd} \]
proof of Weib's theorem:

(1) trivial: \( T(C_r) = : C_r \triangleq C_r \)

(2) want to show:
\[
T(C_r, C_2) = : C_r C_2 \triangleq \angle \# T(C_r, C_2) \tag{3}
\]

either \( T(C_r, C_2) \neq C_r C_2 \) or they differ by

\[
\{ \text{commute} \} \quad \{ \text{form} \}
\]

\[
\{ \text{anticommutate} \} \quad \{ \text{form} \}
\]

\[
T(C_r, C_2) = : C_r C_2 \triangleq \frac{e^{i \theta}}{2} [C_r, C_2] \tag{4}
\]

\[
\text{example: } k_2 > k_2', C_r = q(\hat{k}_r, k_r), C_2 = q^*(\hat{k}_r, k_r)
\]

\[
\therefore \ T(C_r, C_2) = C_r C_2
\]

\[
: C_r C_2 : = : C_2 C_r :
\]

\[
\Rightarrow \ T(C_r, C_2) = : C_r C_2 : = \frac{e^{i \theta}}{2} [C_r, C_2] \tag{5}
\]

\[
\text{example: } k_2 > k_2', C = q(\hat{k}, k), C_2 = q^*(\hat{k}, k)
\]

\[
\therefore \ T(C, C_2) = C C_2
\]

\[
: C C_2 : = : C_2 C :
\]

\[
\Rightarrow \ T(C, C_2) = : C C_2 : = \frac{e^{i \theta}}{2} [C, C_2] \tag{6}
\]

\[
\therefore \ T(C, C_2) = : C C_2 : = \frac{e^{i \theta}}{2} [C, C_2] \tag{7}
\]

\[
\therefore \ T(C, C_2) = : C C_2 : = \angle \# T(C, C_2) \tag{8}
\]

For a non-interacting theory the (anti-)commutator \([C_r, C_2]\)

\[
\text{is just a number whose operators at different times are}
\]

\[
\text{indirectly related}
\]

\[
\therefore \ T(C, C_2) = : C C_2 : = \angle \# T(C, C_2) \tag{9}
\]

\[
\therefore \ T(C, C_2) = : C C_2 : = \angle \# T(C, C_2) \tag{10}
\]
2) Wick's theorem proof for case 2

\[ \text{case 2: } n \text{ odd, } T_{n+1} \text{ non-reducible} \]

\[ \rightarrow \text{terms that again with } T_1, \ldots, T_{n+1} \text{ and end with } \]
\[ \langle \Omega | T(C_1 T_{n+1}) | 0 \rangle - \langle \Omega | T(C_1 T_{n+1}) | 0 \rangle = \mathcal{C}_n T_{n+1} = \text{perm.} \]

again what is missing are terms \( \langle \Omega | T(C_1 T_{n+1}) | 0 \rangle = 0 \)

\[ \rightarrow \text{Wick's theorem proof for case 2} \]

\[ \text{case 3: } n \text{ odd, } T_{n+1} \text{ creation operator} \]

\[ \rightarrow \text{terms in which } T_1, \ldots, T_{n+1} \text{ is again a number} \]
\[ \mathcal{C}_n T_{n+1} = \mathcal{C}_n T_{n+1} : \]
\[ \langle \Omega | T(C_1 T_{n+1}) | 0 \rangle - \langle \Omega | T(C_1 T_{n+1}) | 0 \rangle = \mathcal{C}_n T_{n+1} = 0 \]

\[ \rightarrow \text{Wick's theorem proof for case 2} \]

\[ \text{corresponding arguments can be applied to: } T_1, \ldots, T_{n+1} \text{ together up to } T_{n+1}, \ldots, T_n, T_1, T_2 \]

\[ \text{case 4: } n \text{ even, } T_{n+1} \text{ creation operator} \]

\[ \rightarrow \text{same arguments for: } T_1, \ldots, T_{n+1}, T_n, T_1 \]

\[ \rightarrow \text{Wick's theorem for } n+1 \]
application of Wick's theorem:

For the evaluation of \( p(\tau_1, \tau_2, \ldots, \tau_n) \) (cf. page 23),
we have to calculate objects like

\[
\langle 0| T(\tau_1, \ldots, \tau_n) | 0 \rangle,
\]

where \( T \) is the total ordered term of Wick's theorem. We can write

\[
\langle 0| T(\tau_1, \ldots, \tau_n) | 0 \rangle = \langle 0| T(\tau_1, \tau_2) | 0 \rangle \cdots \langle 0| T(\tau_n, \tau_1) | 0 \rangle
\]

and permutation

in addition:

\[
\langle 0| T[\circ D(\tau_1, \tau_2)] | 0 \rangle = 0,
\]

\[
\langle 0| T[\circ D(\tau_1, \tau_2)] | 0 \rangle = 0
\]

The only object we need is the "propagator" or "two-point function" of the field theory

\[
\langle 0| T(\circ D(\tau_1, \tau_2)] | 0 \rangle = \langle \tau_1, \tau_2 \rangle
\]

It is straightforward to calculate this object. We do not go through
the exercise here. Instead we state the important general result: To calculate whatever matrix element in perturbation theory, one only needs to use Wick's theorem to reduce any expression to a product of two-point functions of
the field theory.

- What one needs is a good understanding of the field theory.
- This motivates the study of field relativistic theories
to which we turn next.