Coordinates

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Cartesian Coordinates

- Cartesian Coordinates, $x, y, z$: Given are three arbitrary unit vectors $\vec{i}_x, \vec{i}_y, \vec{i}_z$, which are pairwise perpendicular to each other and oriented in this order by the "right-hand rule"
- each point in space is determined uniquely by its position vector, pointing from the origin of the coordinate system to the point: $\vec{r} = x\vec{i}_x + y\vec{i}_y + z\vec{i}_z$
- dot product: $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$
- cross product

$$\vec{a} \times \vec{b} = (a_y b_z - a_z b_y)\vec{i}_x + (a_z b_x - a_x b_z)\vec{i}_y + (a_x b_y - a_y b_x)\vec{i}_z$$

- Gradient: $\vec{E} = -\text{grad } V$

$$\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\vec{i}_x + \frac{\partial V}{\partial y}\vec{i}_y + \frac{\partial V}{\partial z}\vec{i}_z\right)$$
Spherical Coordinates (Definition)

- **spherical coordinates,** $r, \theta, \phi$
- determines any point uniquely, **not located on the** $z$ **axis**
- **Relation to Cartesian Coordinates**
  (read them off from the figure!):

$$\vec{r}(r, \theta, \phi) = r(\sin \theta \cos \phi \vec{i}_x + \sin \theta \sin \phi \vec{i}_y + \cos \theta \vec{i}_z)$$

- coordinate ranges: $r > 0$, $\theta \in (0, \pi)$, $\phi \in [0, 2\pi)$

Spherical Coordinates (important formulae)

- each vector, **not pointing into the** $z$-direction, is then uniquely determined by its components with respect to these unit vectors: $\vec{E} = E_r \vec{i}_r + E_\theta \vec{i}_\theta + E_\phi \vec{i}_\phi$.
- these basis vectors depend on $\theta$ and $\phi$!
- surface element for sphere of radius $r$: $d\vec{S} = r^2 \sin \theta \, d\theta \, d\phi \, \vec{i}_r$.
- **Gradient**:

$$\vec{E} = -\nabla V = - \left( \frac{\partial V}{\partial r} \vec{i}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \vec{i}_\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \vec{i}_\phi \right)$$

- vector products: $\vec{i}_r \times \vec{i}_\theta = \vec{i}_\phi$, $\vec{i}_\theta \times \vec{i}_\phi = \vec{i}_r$, $\vec{i}_\phi \times \vec{i}_r = \vec{i}_\theta$.
Cylinder Coordinates (Definition)

- cylinder coordinates, $\rho$, $\phi$, $z$
- determines any point uniquely, not located on the $z$ axis
- Relation to Cartesian Coordinates (read them off from the figure!):
  \[
  \vec{r}(\rho, \phi, z) = \rho \cos \phi \vec{i}_x + \rho \sin \phi \vec{i}_y + z \vec{i}_z
  \]
- coordinate ranges: $\rho > 0$, $\phi \in [0, 2\pi)$, $z \in \mathbb{R}$

Cylinder Coordinates (important formulae I)

- each vector, not pointing into the $z$-direction, is then uniquely determined by its components with respect to these unit vectors: $\vec{E} = E_\rho \vec{i}_\rho + E_\phi \vec{i}_\phi + E_z \vec{i}_z$.
- $\vec{i}_\rho$ and $\vec{i}_\phi$ depend on $\phi$!
- surface element for cylinder envelope $\rho = \text{const}$: $d\vec{S} = \rho \, d\phi \, dz \vec{i}_\rho$.
- surface element for upper cylinder cap: $d\vec{S} = \rho \, d\rho \, d\phi \vec{i}_z$.

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Cylinder Coordinates (important formulae II)

\[ \vec{i}_\rho = \cos \phi \vec{i}_x + \sin \phi \vec{i}_y, \]
\[ \vec{i}_\phi = -\sin \phi \vec{i}_x + \cos \phi \vec{i}_y, \]
\[ \vec{i}_z = \vec{i}_z \]

\( \vec{E} = -\nabla V = - \left( \frac{\partial V}{\partial \rho} \vec{i}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \vec{i}_\phi + \frac{\partial V}{\partial z} \vec{i}_z \right) \)

▶ vector products: \( \vec{i}_\rho \times \vec{i}_\phi = \vec{i}_z, \vec{i}_\phi \times \vec{i}_z = \vec{i}_\rho, \vec{i}_z \times \vec{i}_\rho = \vec{i}_\phi. \)