1. (25 points) A charge $Q$ is uniformly spread along the $x$-axis (from $x = -a$ to $x = a$).

(a) Determine the electric field, $\vec{E}(x, y)$ of this charge distribution.

\[
\vec{E} = \frac{1}{8 \pi \varepsilon_0 a} \int_{-a}^{a} dx' \frac{(x-x') \hat{i}_x + y \hat{i}_y}{[(x-x')^2 + y^2]^{3/2}} \tag{5}
\]

\[
= \frac{Q}{8 \pi \varepsilon_0 a} \left[ \frac{1}{\sqrt{(x-a)^2 + y^2}} - \frac{1}{\sqrt{(x+a)^2 + y^2}} \right] \hat{i}_x \tag{5}
\]

\[
+ \left[ \frac{x+a}{\sqrt{(x+a)^2 + y^2}} - \frac{x-a}{\sqrt{(x-a)^2 + y^2}} \right] \frac{\hat{i}_y}{y} \tag{5}
\]

(b) What is the electric force on a test charge, $q$, located on the $y$ axis at $y = b$?

\[
\vec{F} = q \frac{\vec{E}}{y = b} = \frac{Q}{4 \pi \varepsilon_0} \left[ \frac{q}{\sqrt{b^2 + x^2}} \right] \hat{i}_x \tag{5}
\]
2. (25 points) A sphere of radius, R, is charged with a charge distribution such that the charge density \( \rho(r) = \alpha r \), where \( \alpha \) = const.

(a) Determine the electric field, \( \vec{E} \), everywhere.

\[
\oint d\vec{S} \cdot \vec{E} = 4\pi r^2 E_r = \frac{\rho_{\text{inside}}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \alpha \int_0^r r^4 dr \quad \text{for} \quad r < R
\]

\[
= \int_0^r \frac{\alpha r^2}{\varepsilon_0} \, dr \quad \text{for} \quad r < R
\]

\[
\Rightarrow E_r = \begin{cases} \frac{\alpha r^2}{4\varepsilon_0} & \text{for} \quad r < R \\ \frac{\alpha r_4}{4\varepsilon_0} & \text{for} \quad r > R \end{cases}
\]

(b) Calculate the potential difference between the center of the sphere and a point outside (at a distance \( r > R \) from the center).

\[
E_r = -\frac{dV}{dr} \Rightarrow V(r) = \begin{cases} \frac{-\alpha r^3}{12\varepsilon_0} + C & \text{for} \quad r < R \\ \frac{\alpha r_4}{4\varepsilon_0} & \text{for} \quad r > R \end{cases}
\]

\[
V(r) = \begin{cases} \frac{-\alpha r^3}{12\varepsilon_0} & \text{for} \quad r < R \\ \frac{\alpha r_4}{4\varepsilon_0} & \text{for} \quad r > R \end{cases}
\]

\[
V(0) - V(R) = \frac{\alpha R^3}{3\varepsilon_0} - \frac{\alpha R^4}{4\varepsilon_0}
\]
3. (25 points) Two parallel very large conducting plates are connected to a battery with voltage, $V$, for a long time. Both have a hole in the middle, and at $t = 0$ a charged particle with mass $m$, and charge, $q > 0$, enters the hole of the lower plate with negligible velocity. Outside of the plates is a homogeneous magnetic field. Gravity can be neglected.

(a) What is the particle's velocity when it leaves the capacitor at the upper plate?

\[ \frac{m}{2} v^2 + qV = \text{const} \Rightarrow qV = \frac{m}{2} v^2 \]

\[ v = \sqrt{\frac{2qV}{m}} \]  \hspace{1cm} (5)

(b) Determine the distance, $a$, of the point where the particle hits the upper plate again.

\[ \frac{m}{q} \frac{v^2}{R} = qvB \Rightarrow R = \frac{mv}{qB} \]  \hspace{1cm} (5)

\[ x_{\text{end}} = 2R = \frac{2m}{qB} \sqrt{\frac{2qV}{m}} = \frac{2}{B} \sqrt{\frac{2mV}{q}} \]  \hspace{1cm} (5)
4. (25 points) The circuit is hooked up to the battery as shown for a very long time. 
(a) What are the currents through each resistor?

\begin{align*}
\text{I} : & \quad -v + R_1 i_1 + R_3 i_3 = 0 \\
\text{II} : & \quad -R_1 i_1 - R_2 i_2 = 0 \\
\text{III} : & \quad -R_3 i_3 + \frac{Q}{C} = 0 \\
\text{IV} : & \quad i_1 + i_2 = i_3
\end{align*}

\begin{align*}
R_1 i_1 + R_3 (i_1 + i_2) = v & \quad \Rightarrow i_2 = \frac{R_1}{R_2} i_1 \\
(R_1 + \frac{R_1 R_3}{R_2}) i_1 = v & \Rightarrow i_1 = \frac{R_2 v}{(R_1 + R_3) R_2 + R_1 R_3} \\
\dot{i}_2 = \frac{R_1 v}{(R_1 + R_3) R_2 + R_1 R_3} & \Rightarrow \dot{i}_3 = \frac{(R_1 + R_2) v}{(R_1 + R_3) R_2 + R_1 R_3}
\end{align*}

(b) What is the voltage across the capacitor? What is the charge at its positively charged plate?

\[
\frac{\dot{Q}}{C} = v_c = R_3 \dot{i}_3 = \frac{R_3 (R_1 + R_2) v}{(R_1 + R_3) R_2 + R_1 R_3}
\]

**Hint:** Label all currents and the signs of charges at the capacitor in the circuit diagram!
5. (25 points) A thin wire in the $xy$ plane of a Cartesian coordinate system is shaped as shown in the figure: an infinitely long piece lies along the $x$ axis from $x \to -\infty$ to $x = 0$, then a piece is shaped as a quarter of a circle of radius, $R$. Finally, another infinitely long piece is placed at $x = R$ parallel to the $y$ axis from $y = R$ to $y \to \infty$.

Calculate the magnetic field, $\vec{B}$, at the point $P$, located at $x = R, y = z = 0$, which is the center of the quarter circle!

$$\vec{B}_P = \frac{\mu_0 i}{4\pi} \int_0^{\pi/2} d\phi \frac{r^2}{\sqrt{r^2 - z^2}}$$

$C_1 + C_3$: no contribution since $d\vec{z}_1 \parallel \vec{r} - \vec{z}_1$ \hspace{1cm} ⑤

$C_2$:

$$\vec{z}_2 = (R - R\cos \phi) \hat{i}_x + R\sin \phi \hat{i}_y \hspace{1cm} \vec{r}^2 = R^2 \hat{i}_x$$ ⑤

$$d\vec{z}_2 = d\phi \left[ R \sin \phi \hat{i}_x + R \cos \phi \hat{i}_y \right]$$ ⑤

$$d\vec{r}_x \times (\vec{r} - \vec{z}_1) = d\phi \left[ R \sin \phi \hat{i}_x + R \cos \phi \hat{i}_y \right] \times \left[ R \cos \phi \hat{i}_x - R \sin \phi \hat{i}_y \right]$$

$$= d\phi \left[ R^2 \hat{i}_x - \hat{i}_z \right]$$

$$\vec{B}_P = -\frac{\mu_0 i}{4\pi} \int_0^{\pi/2} d\phi \frac{R^2}{\sqrt{R^2 - z^2}} = -\frac{\mu_0 i}{8R} \hat{i}_z \hspace{1cm} ⑤$$
6. (25 points) A circular wire of radius, \( r \), of cross-sectional area, \( A \), is made of a material with resistivity \( \rho \). For a very long time, it rotates around one of its diameters with constant angular velocity, \( \omega \). A homogeneous magnetic field, \( B \), is pointing perpendicular to the rotation axis.

(a) Calculate the current induced in the wire.

\[
R \ i = - \frac{d\Phi}{dt} = - \frac{d}{dt} \left[ \pi r^2 B \omega \ (ut) \right] \tag{10}
\]

\[
i = + \frac{\pi r^2 B \omega}{R} \ \sin \ (\omega t) \tag{9}
\]

\[
R = \frac{\rho}{\sigma} = \frac{2\pi \ell A}{A} \tag{5}
\]

(b) How much energy is used during one period of rotation \( T = 2\pi/\omega \)?

\[
P(t) = R[i(t)]^2 = \frac{\pi^2 \ell^4 B^2 \omega^2}{R^2} \cdot [\sin(\omega t)]^2 \tag{3}
\]

\[
E_{\text{total}} = \int_0^T dt \ P(t) = \frac{\pi^3 \ell^4 B^2 \omega}{R^2} \tag{2}
\]
7. (25 points) (a) In the circuit shown in the figure, the switch has been in position A for a long time.

(a) What is the charge at the upper plate of the capacitor?

\[-V + \frac{Q_0}{C} = 0 \Rightarrow Q_0 = CV\]  

(b) At \( t = 0 \) the switch is set into position B. Calculate the charge, \( Q(t) \), at the upper plate of the capacitor as a function of time.

\[
\frac{Q}{C} = +L \frac{d^2Q}{dt^2} \Rightarrow \dot{Q} = -\frac{1}{LC} Q \]  

\[
Q(0) = Q_0 \quad \Rightarrow Q = Q_0 \cos(\omega t) \]  

(c) What is the current, \( i(t) \), through the coil, as a function of time (for \( t > 0 \))?

\[
i(t) = \dot{Q} = A\omega \sin(\omega t) \]
8. (25 points) (a) Suppose the switch has been open for a very long time. At \( t = 0 \) it is closed. Calculate the currents through the resistors as a function of time! All self-inductances can be neglected.

\[
\begin{align*}
I: & \quad -V + R_1 \frac{d}{dt} i_1 = 0 \quad \Rightarrow \quad i_1 = \frac{V}{R_1} \quad \text{(5)} \\
\text{\(i_1\) jumps from 0 to steady-state value.} \\
II: & \quad -V + R_2 \frac{d}{dt} i_2 + \frac{Q}{C} = 0 \quad \Rightarrow \quad i_2 = \frac{Q}{C} \quad \text{(5)} \\
& \Rightarrow \quad \dot{Q} + \frac{Q}{R_2 C} - \frac{V}{R_2} = 0 \quad \text{(5)} \\
& \quad Q(t) = A \exp \left(-\frac{t}{\tau_2}\right) + CV \quad \text{\(Q(0) = 0 \Rightarrow \) \(Q(t) = -CV \exp \left(-\frac{t}{\tau_2}\right)\)} \quad \text{(5)} \\
& \quad Q(t) = CV \left[1 - \exp \left(-\frac{t}{\tau_2}\right)\right] \quad \text{(5)}
\end{align*}
\]

(b) Show that after a long time \( (t \to \infty) \) the currents reach the values to be expected from a steady-state situation.

\[
i_1 \text{\(\to 0\)} \quad \text{immediately to the steady-state value.} \quad \text{(5)}
\]

\[
i_2(t) = \frac{V}{R_2} \exp \left(-\frac{t}{\tau_2}\right)
\]

The uncharged capacitor is like a short-circuit in the very first moment. Thus the current decays to 0, which is the steady-state value.