5. The magnetic field

5.1 Preliminary ideas about sources of magnetic fields

So far we have looked only at electrostatic fields, i.e., electric fields which are independent of time. Thus any force due to charge is a property of matter, namely to carry an electric charge. On the micro level, however, there is another way particles like electrons could cancel out electric charge and carry a current. This is the case of a magnet, a material composed of electric charge and the electric field by such a point charge, \( q \), is given by Coulomb's law, which reads as our results

\[
E = \frac{q}{4\pi\varepsilon_0} \frac{r}{r^3}
\]

when we assume the charge to be located at the origin of the coordinate system.

Now imagine the dipole which consists of two new elementary charges which are equal in magnitude but opposite in sign:

\[
E = \frac{\alpha}{4\pi\varepsilon_0} \left[ \frac{r^2 - \frac{d^2}{2} \hat{y}}{\left| r^2 - \frac{d^2}{2} \hat{y} \right|^3} - \frac{r^2 + \frac{d^2}{2} \hat{y}}{\left| r^2 + \frac{d^2}{2} \hat{y} \right|^3} \right]
\]

where \( \alpha \) is the dipole moment.
Now suppose we only are interested in cases of $r > a$. To simplify this field for these limiting cases, we write the electric potential

\[ V(r) = \frac{Q}{4\pi \epsilon_0} \left( \frac{1}{r - \frac{a}{2} \cos \phi} - \frac{1}{r + \frac{a}{2} \cos \phi} \right) \]

Now we have

\[ (r^2 - \frac{a}{2} \cos \phi)^2 = r^2 + \left(y - \frac{a}{2}\right)^2 + \frac{a^2}{4} \]

\[ \approx r^2 + \frac{a^2}{4} \left(1 - \frac{dy}{r^2} + \frac{d^2}{4r^4} \right) \]

and hence because $\frac{d}{r} \ll 1$:

\[ \frac{1}{r - \frac{a}{2} \cos \phi} \approx \frac{1}{r} \left[ 1 + \frac{1}{2} \frac{dy}{r^2} + O\left(\frac{d^2}{r^4}\right) \right] \]

In the same way:

\[ \frac{1}{r + \frac{a}{2} \cos \phi} \approx \frac{1}{r} \left[ 1 - \frac{1}{2} \frac{dy}{r^2} + O\left(\frac{d^2}{r^4}\right) \right] \]

\[ \Rightarrow \quad V(r) \approx \frac{Q}{4\pi \epsilon_0} \frac{dy}{r^3} \]
One defines
\[ \mathbf{P} = \mathbf{Q} \mathbf{r} \]

as the electric dipole moment. Then the potential of a dipole reads
\[ V(\mathbf{r}) = -\frac{\mathbf{P} \cdot \mathbf{r}}{4\pi \varepsilon_0 r^3} \]

This expression becomes exact when we let \( \mathbf{Q} \to 0 \) but make \( \mathbf{Q} \to \infty \) so that \( \mathbf{Q} \cdot \mathbf{r} = |\mathbf{P}| = \) const. One could imagine that have a pointlike particles with a change in space. However, in the sense of the approximation is known. It's still a useful concept.

Now we can calculate the electric field
\[ \mathbf{E} = -\nabla V = -\frac{1}{4\pi \varepsilon_0} \left[ \frac{\mathbf{P} \cdot \mathbf{r}}{r^3} - \frac{3}{r^5} (\mathbf{P} \cdot \hat{r}) \right] \]

\[ \Rightarrow \mathbf{E} = \frac{1}{4\pi \varepsilon_0} \left[ \frac{3(\mathbf{P} \cdot \hat{r})}{r^5} \mathbf{r} - \frac{\mathbf{P}}{r^3} \right] \]

Here's what we need:

\( \mathbf{E} \) is a sphere with center in the origin the total charge inside is always 0.

**Solution:**

We use spherical coordinates with \( \mathbf{P} \) defining the polar axis (let's denote it our standard spherical coordinates).
This is a page with mathematical equations and text. The text appears to be discussing physics or engineering concepts, possibly related to electrical circuits or waveforms. The equations include symbols and variables typically found in technical discussions. The page contains a mix of algebraic expressions and equations, with some text explaining the context or derivation. The handwriting is slightly slanted, which might make some parts of the text a bit challenging to read without clarity. However, the main focus seems to be on the equations and their relationships, which are written in a formal and technical manner.
Thus, this is nothing like a simple magnetic charge. Some physicists have thought about such magnetic poles, but so far nobody has ever seen one.

In a permanent magnet, on the other hand, we have a magnetic dipole moment, and there is no need to make of a permanent magnet.

We shall come back to the question of other sources of magnetic fields. Most probably the fact that electric currents always create a magnetic field, which we call $B$.

9.2 Magnetic Force on a Point Charge

Now we ask what a more general question. What are the forces magnetic fields exert on particles. The most easy case is a charged particle without magnetic dipole moment. It turns out that a magnetic field only acts on moving charges, and the law is quite simple compared to the electric forces:

$$F_m = q \times \vec{B}(r)$$

where $\vec{r}$ is the position of the particle, $q$ is its charge, and $\vec{v}$ is its velocity.

This law is a result of many experiments and cannot be derived from more fundamental laws.
Now we can determine the unit of the magnetic field:

\[
\vec{B} = \frac{\vec{F}}{\vec{E} \cdot \vec{J}} = \frac{N}{C \frac{m}{s}} = \frac{N}{C m} = 1 \text{ T (Tesla)}
\]

\[
= \frac{1 \text{ Wb}}{m^2} \quad \text{(Wb = Weber, m² = square meter)}
\]

9.3 Motion of a particle in a constant B field

Take \( \vec{B} = \text{const.} \), then the equation of motion reads

\[
m \frac{d\vec{v}}{dt} = -\vec{F}
\]

\[
m \frac{d\vec{B}}{dt} = q \vec{v} \times \vec{B}
\]

\[
m \frac{d\vec{B}}{dt} = 0 \quad \text{since \( \vec{B} = \text{const.} \)}
\]

But \( \vec{B} \) is a constant, the \( \vec{B} \)-field is negligible and disappears. The

Let's point the \( \vec{B} \)-field in negative z-direction. Then

we have as in the book:

\[
\vec{B} = B \hat{z}
\]

(\( \vec{B} \) pointing away from you)

\[0 \hat{x} - B \hat{z}
\]

\[
B = -B \vec{e}_z
\]

In components our EOM reads

\[
\frac{d\vec{v}}{dt} = \frac{4}{m} \left( v_x \vec{e}_x + v_y \vec{e}_y + v_z \vec{e}_z \right) (-B \vec{e}_z)
\]

\[
= \frac{4}{m} \left( v_x B \vec{e}_y - B v_y \vec{e}_x \right)
\]
\[ \frac{dW_x}{dt} = -\frac{4B}{m} v_y \]  

(1)

\[ \frac{dW_y}{dt} = \frac{4B}{m} x \]  

(2)

To solve these equations we take the time derivative of (1) and use (2) on the RHS:

\[ \frac{d^2 W_x}{dt^2} = -\frac{4B}{m} \frac{dW_y}{dt} = -\left(\frac{4B}{m}\right)^2 v_x \]

The most general solution of these equations is:

\[ v_x = a \sin(\omega t + \phi) \]

with \( W = \frac{4B}{m} \) and \( a \) and \( \phi \) as arbitrary constants which we have to determine from the initial conditions which we have to determine from the initial conditions.

From (1) we find:

\[ \frac{dW_y}{dt} = -\frac{1}{W} \frac{dW_x}{dt} = \frac{a \omega}{W} (\omega t + \phi_0) \]

From this we find by one more integration:

\[ v_x = \frac{dx}{dt} = \frac{a \omega}{W} (\omega t + \phi_0) + b \]

\[ y = \frac{a}{W} \sin(\omega t + \phi_0) + c \]
Suppose now the situation that at \( t=0 \) we have 
\[ x=0, \quad y=v_0 \]
\[ \dot{x}=0, \quad \dot{y}=v_0 \]

Then \( \phi_0 = 0, \quad \alpha = \frac{v_0}{2} \)
\[ x = \frac{v_0}{2} \sin(\omega t) + x_0 - \frac{v_0}{\omega} \]
\[ y = \frac{v_0}{2} \cos(\omega t) + y_0 \]

The particle moves around a circle with the "cyclotron frequency" \( \omega = \frac{qB}{m} \), which is independent of the initial conditions.\[ W = \frac{m}{q} \]

This leads to the idea of constructing the cyclotron which, finally led to a Nobel prize for him!\[ W \] shall treat the principle of the cyclotron in the next section:

Note that the line along the \( z \) direction is \( 0 \):
\[ m \ddot{z} = m \frac{dv_z}{dt} = 0 \]
\[ \Rightarrow v_z = \text{const} \Rightarrow z = v_{0z} t + z_0 \]

The motion of the particle is thus a spiral in \( \theta \). For \( v_{0z} = 0 \) it's a circle in the plane \( t = z_0 = \text{const} \) parallel to the \( xy \) plane.
9.4 Crossed $E$ and $B^2$ Fields

Obviously, if a charge $q$ moves through a region where both an electric and magnetic field are present, it feels a force

$$ F = q \left( E + \vec{v} \times \vec{B} \right) $$

which is the complete version of the Lorentz force. As an example, we look at a driven, accelerating physicist who to sort out charged particles at a given speed.

A particle entering $S_1$ can also leave $S_2$, if we know defects of. Now we have

$$ F = q \left( E^2 + \vec{v} \times \vec{B} \right) $$

In our coordinate system we have

$$ E_x = E^2, \quad B_z = B \vec{e}_z, \quad \vec{v} = v \vec{e}_x $$

and thus

$$ \vec{F} = q \left( E \vec{e}_y + v B \vec{e}_x \times \vec{e}_t \right) $$

$$ = q \left( E - v B \right) \vec{e}_y $$

You can see the right-hand rule to verify the direction of the magnetic force.

Thus on any particles, breaded

$$ E - v B = 0 \Rightarrow v = \frac{E}{B} $$

go through $S_2$. 

\[ \text{\textcopyright 1983} \]
9.5 Magnetic force on a current-carrying wire

We can determine the force on a wire with a current \( \mathbf{i} \) passing through it by the usual trick to first determine the force element \( d\mathbf{F} \) on a small current element \( d\mathbf{i} \) on the wire.

![Diagram of a current-carrying wire]

Now we have the current density vector

\[
\mathbf{j} = -e \mathbf{m} \mathbf{V} \left( \mathbf{r} \right) \text{ and its unit volume}
\]

which runs in the direction of the current. As we have learnt in the previous chapter, how the magnetic force on the little volume element \( dV \)

is

\[
d\mathbf{F} = d\mathbf{A} \times \mathbf{B} \left( \mathbf{r} \right)
\]

\[
= -e \mathbf{m} \mathbf{V} \left( \mathbf{r} \right) dV \mathbf{j} \left( \mathbf{r} \right) \times \mathbf{B} \left( \mathbf{r} \right)
\]

\[
= dV \mathbf{j} \left( \mathbf{r} \right) \times \mathbf{B} \left( \mathbf{r} \right)
\]

The total force is then the integral

\[
\mathbf{F} = \int dV \mathbf{j} \left( \mathbf{r} \right) \times \mathbf{B} \left( \mathbf{r} \right)
\]

For a long wire we can assume that \( \mathbf{j} \) is constant along the wire (in the steady state) and thus have by

the wire (in the steady state) and thus have by

\[
\mathbf{j} = \frac{I}{A}
\]

\[
\mathbf{F} = \frac{I}{A} \int \mathbf{B} \left( \mathbf{r} \right) dA
\]
In a direction of \( \mathbf{j} \). As we learnt before, this direction is determined by the rule that the current density points along the voltage drop.

\[
\mathbf{F} = \int_{\text{win}} \mathbf{J} \times \mathbf{B}
\]

\[
= \int_{\text{win}} 
\frac{d\mathbf{A}}{dt} \times \mathbf{B}
\]

\[
= \int_{\text{win}} \mathbf{i} ds \times \mathbf{n} \times \mathbf{B}
\]

We sometimes write show
\[
ds \mathbf{n} = ds \mathbf{n}
\]

The our law reads
\[
\mathbf{F} = \mathbf{i} \int_{\text{win}} ds \mathbf{n} \times \mathbf{B} (\mathbf{r})
\]

In an inhomogeneous \( \mathbf{B} \) field and a straight wire we have
\[
\mathbf{F} = \mathbf{i} \int_{\text{win}} ds \mathbf{n} \times \mathbf{B} = -\mathbf{i} \mathbf{B} \times \int_{\text{win}} ds \mathbf{n}
\]

\[
\mathbf{F} = +\mathbf{i} \mathbf{l} \times \mathbf{B}
\]

3.6 Hall Effect

See problems Chapt. 9
10.1 Biot–Savart's law

In 1820 the Danish physicist Hans Christian Ørsted discovered that along a current conducting wire, a magnetic field is created with a magnetic field proportional to the current and the distance from the wire. The direction is given by the right-hand rule:

\[ \mathbf{B} = \frac{\mathbf{i} \times \mathbf{B}}{\mathbf{a}}, \]

The direction of the current is as usual, that of \( \mathbf{i} \).

Almost at the same time Ampère postulated that the magnetic field \( \mathbf{B} \) can be calculated by the rule that a current element \( i \, ds \) produces the field element

\[ \mathbf{dB} = \frac{\mu_0}{4\pi} \frac{i (ds \times \mathbf{r})}{r^3} \]

where \( \mathbf{r} \) is the vector pointing from the current element to the point at which we want to calculate \( \mathbf{B} \). For this reason, the current element \( i \, ds \) is always directed in the direction of the current in our assumed sense.

It is fixed to remember in which we do the following:

\[ C : \mathbf{z} = \mathbf{z}'(t) \]

denoting the parametrization of the wire which is chosen such that

\[ ds^2 = dt^2 + ds'^2 \]

\[ \frac{dz}{dt} \] always points in the direction of the current in our assumed sense.
Then Ampère's law reads

\[ \mathbf{B}(r^2) = \frac{-i}{2\mu_0} \int \frac{r'^2}{|r'^2 - r|^3} \, dx'^2 \times \mathbf{e}_x \]

oriented such that the loop is traversed for increasing \( r \).

\[ x = r - r' \]

Now let's calculate the \( \mathbf{B} \) field of an infinitely long wire along the \( x \)-direction (as in the book):

The wire is chosen

\[ r^2 = x'^2 + y'^2 + z'^2 \]

Further

\[ \mathbf{B}(r^2) = \frac{\mu_0 i}{2\pi} \int_0^{\infty} dx'^2 \times \mathbf{e}_x \]

\[ \frac{(x-x') i_x + y i_y + z i_z}{[(x-x')^2 + y^2 + z^2]^{3/2}} \]
\[ \mathbf{B}(r^2) = \frac{\mu_0 c}{4\pi} \int_{-\infty}^{\infty} dx \left( \frac{y^2 c_2^2 - z^2 c_2^2}{(x-x')^2 + y^2 + z^2} \right)^{3/2} \]

\[ \mathbf{B}(r^2) = \frac{\mu_0 c}{4\pi} \frac{x - x'}{(y^2 + z^2) \left[ (x-x')^2 + y^2 + z^2 \right]^{3/2}} \]

\[ \mathbf{B}(r^2) = \frac{\mu_0 c}{2\pi} \frac{y c_2^2 - z c_2^2}{y^2 + z^2} \]

\[ \mathbf{B}(r^2) = \frac{1}{2\pi} \frac{y c_2^2 - z c_2^2}{\sqrt{y^2 + z^2}} \]

\[ \mathbf{B}(r^2) = \frac{\mu_0 c}{2\pi} \frac{1}{\sqrt{y^2 + z^2}} \]

As expected from symmetry, \( |\mathbf{B}| \) is independent of \( z \). That we also calculated the right direction becomes clear when we look on the \( yz \)-plane.

In cylindrical coordinates with the \( x \) axis as cylindrical axis, we have

\[ B(r^2) = \omega r \phi c_2^2 + \omega y c_2^2 = \frac{y c_2^2}{\sqrt{y^2 + z^2}} - \frac{z c_2^2}{\sqrt{y^2 + z^2}} \]

\( \phi \) is undefined in the case of \( r^2 = g c_2^2 + t c_2^2 \)

and hence the case of \( r^2 = g c_2^2 + t c_2^2 \)

\[ \mathbf{B}(r^2) = \frac{\mu_0 c}{2\pi} \phi (r^2) \]
Thus Ampère's law proves the right result for a long straight wire according to Biot and Savart.

10. 2 Force on two current-carrying wires and the definition of the Ampère

Suppose we have the wire of the previous section and put another wire with current \( i_2 \) in the direction \( d \). According to the previous result, if we have a wire of length \( l_c \) and current \( i_c \), then is the magnetic force received by this second wire:

\[
\vec{F} = i_c \vec{B} \times \vec{i}_c
\]

From the previous section we have \( \vec{B} \) along the second wire:

\[
\vec{F} = \frac{\mu_0 i_2}{2\pi d} \vec{e}_z \times \vec{i}_c = -\frac{\mu_0 i_2 l}{2\pi d} \vec{e}_z
\]

The answer is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section and placed 1 m apart in vacuum, would produce between these conductors a force equal to \( 2 \times 10^{-2} \) newtons per meter of length.

According to the formula we have just derived, this amounts to the definition of the permeability of the vacuum:

\[
\mu_0 = \frac{\mu_0 \cdot 1 \text{ m}^2}{2\pi} \Rightarrow \mu_0 = 4\pi \cdot 10^{-7} \frac{\text{N}}{\text{A}^2}
\]
Next we establish the magnetic field of a current-carrying loop, a circle of radius $a$ in the $xy$ plane, along the $z$ axis:

$$\vec{r} = a \hat{e}_z$$

$$d\vec{r} = a \hat{e}_r \frac{d\phi}{d\theta}$$

$$\vec{B} = \frac{\mu_0}{4\pi} \int_0^{2\pi} d\phi \frac{a^2 \hat{e}_r \times \vec{r}}{(r^2 + a^2)^{3/2}}$$

$$\vec{r} = a \hat{e}_z \times (r - r') = \vec{r}' \times \left[ r \hat{e}_z - a \hat{e}_z \right]$$

$$= \vec{r}' \times \hat{e}_z$$

$$\int_0^{2\pi} d\phi \frac{a^2 \hat{e}_r \times \vec{r}}{(r^2 + a^2)^{3/2}} = 0$$

We find

$$\vec{B}(r, \theta) = \frac{\mu_0}{4\pi} \int_0^{2\pi} d\phi \frac{a^2 \hat{e}_r \times \vec{r}}{(r^2 + a^2)^{3/2}} = \frac{\mu_0 I_0}{2} \frac{a^2}{(r^2 + a^2)^{3/2}} \hat{e}_r$$

For $r > a$ we obtain

$$\vec{B}(r, \theta) = \frac{\mu_0 I_0}{2\pi} \frac{a^2}{r^3}$$
Where the magnetic moment of the loop is defined by
\[ \mathbf{m} = \mathbf{A} \times \mathbf{i} \]

10.4 Ampère's Circuital Law

Now we look at an analogous law for \( \mathbf{B} \) as we found in terms of Gauss's law for \( \mathbf{E} \). Here we shall not prove it in full generality but shall stick to the case of an infinitely long wire.

The statement is:

For any closed curve \( C \) encircling the wire, we have
\[ \oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 \oint_C d\mathbf{S} \times \mathbf{j} \]

where \( \mathbf{j} \) is the total current passing through any surface with \( C \) as boundary. The surface elements must be oriented in relation to the orientation of the curve \( C \) correspondingly.

The statement is in fact valid for all stationary currents.

The statement is for loop, valid for all stationary currents: \( \mathbf{B} \) is:
\[ \oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 \int d\mathbf{S} \times \mathbf{j} \]

\( \mathbf{B} \) field is not (a conservative field!)
It is important to note that this is only a consistency statement if there is a conserved. Suppose we choose another surface \( S' \) with the same boundary area: \( \partial S' = \partial S \). Then we must have

\[
\oint_{\partial S} \mathbf{B} \cdot d\mathbf{r} = \iiint_{V} \nabla \times \mathbf{B} \cdot d\mathbf{V}
\]

(b) \[
\oint_{\partial S'} \mathbf{B} \cdot d\mathbf{r} = \iiint_{V} \nabla \times \mathbf{B} \cdot d\mathbf{V}
\]

Because \( \partial S' = \partial S \)

Now \( S - S' \) is a closed surface, "- S" meaning the same

surface as \( S' \) but with the opposite orientation. From the points

laws we in [b] from (11) and (12) that

\[
\oint_{\partial S} \mathbf{B} \cdot d\mathbf{r} = 0
\]

\( S - S' \)

Since on the other hand we can draw any closed surface

and draw an arbitrary closed arrow on it (we must have

\[
\oint_{\partial S} \mathbf{B} \cdot d\mathbf{r} = 0
\]

\( S \)

for any closed surface. As we know from the previous

chapters, that's precisely stating the conservation of electric charge.
steady-steady currents

10.5 Proof of Ampère's circuital law for the B field of an infinite wire

(a) Show that the wire carries an on-coming current

\[ \mathbf{B} = \frac{\mu_0}{2\pi} \frac{\mathbf{J}_0 \times \mathbf{r}}{r^2} \]

\[ B_z = \frac{\mu_0}{2\pi} \int \frac{J_0 \, dr}{r} \Rightarrow \frac{dr^2}{dr} = a \, \epsilon \phi \]

\[ B_z[\phi(\mathbf{r})] = \frac{\mu_0}{2\pi} \frac{\epsilon \phi}{a^2} \]

\[ \oint \mathbf{B} \cdot d\mathbf{r} = \oint \mathbf{dA} \cdot \frac{\mu_0 \epsilon \phi}{2\pi} \frac{i}{a} = \mu_0 i \quad \square \]

(b) Arbitrary cross in the plane \( \perp \) to wire containing current

\[ \text{Because } \epsilon_\phi, \frac{\partial \phi}{\partial r} = 1 \]
We can approximate this path closely by precisely piecewise chordal sectors of a circle with the cone as center, connected by straight radial lines:

Since all the radial segments give always a contribution

\[ \frac{\theta_0}{2\pi} \cdot \Delta \Phi \]

and the radial lines for \( \alpha \) being \( \alpha = \theta_0 \) and \( B = \theta \) and \( \theta_3 = 0 \), we again find after summing over all segments:

\[ \oint \Phi \cdot B(r) = \rho_0 \cdot \theta_0 \]

(c) For a non-planar curve containing the cone in its interior

Then we can do a similar experiment. We approximate the path by arcs of circles in planes \( \perp \) to the wire connected by radial lines.
The pieces $\vec{\partial} x$ and $\vec{\partial} y$ directions give $0$ in the integral in case $\partial x^2 - B y^2$. The circular segments seem to go to $0$ as in case (b).

(a) Curves not containing the core

This can be approximated by segments and straight radial and lines $\perp$ to the core as well:

$$
\int_D \vec{\partial} x \cdot \vec{\partial} y = \frac{100}{2\pi} \Delta y
$$

$$
\int_A \vec{\partial} x \cdot \vec{B}(r) = -\frac{100}{2\pi} \Delta \varphi
$$

And thus

$$
\oint_C \vec{\partial} x \cdot \vec{B}(r) = 0
$$

This is a rather simple proof for Ampère's law for the $\vec{B}$-field of an in-finite current-carrying core, but much work on his circular law is the basis for the core plays between currents and $\vec{B}$ fields. Because it is true for all surfaces $\Sigma$ with arbitrary boundaries $\partial \Sigma$. 

\[\text{Diagram with a vector field}\]
it can be much a local law connecting $B$ and $j$. We won't go on to this in this lecture, but rather use it for some very simple but rather important situations.

10.6 Superposition

If we have more than one current-carrying wire, the total $B$ field is given by the sum of the $B$ fields. This can be seen with Ampère's circuital law as follows: we can use Ampère's circuital law as follows: we can use the right-hand rule to show that the total current $I$ is the sum of the currents $i_1, i_2, \ldots, i_n$ as we sum the contributions to the magnetic field $B$ as we sum the contributions to the magnetic field $B$ around a surface $S$.

Drawing this on a plane makes it easier to see, but it is valid for any surface, not only plane ones.

Since the paths inside the surface are made through twice in opposite directions, the contributions along these cancel out. Thus, we can write...
\[ \int\frac{d^2 B}{dS} = \int\frac{d^2 B}{dS_1} + \int\frac{d^2 B}{dS_2} + \ldots + \int\frac{d^2 B}{dS_m} \]

\[ = \sum_{k=1}^{m} \int\frac{d^2 \vec{B}}{dS_k} = \int_0^j \sum_{k=1}^{m} i_k \]

On the other hand, if we denote the \( \vec{B} \)-field inside each m-shell by \( \vec{B}_k \), we have

\[ \int\frac{d^2 \vec{B}}{dS_k} = \begin{cases} 0 & \text{if } k \neq k' \\ i_k & \text{if } k = k' \end{cases} \]

But this also holds true for the boundary \( dS \):

\[ \int\frac{d^2 \vec{B}}{dS} = i_k \]

and likewise

\[ \int\frac{d^2 l}{dS} \vec{B}_k = \int_0^j \sum_{k=1}^{m} i_k \]

Closed surface

So we find, since this is called the anyonic \( \vec{B} \)-field

\[ \vec{B}_m = \sum_{k=1}^{m} \vec{B}_k \]
10.7 Applications of Ampère's Law

(a) The long solenoidal coil

Assumption: \( \mathbf{B} = B_x \mathbf{\hat{x}} = \text{const.} \) inside the cylinder, while \( B = 0 \) outside. Then we choose the rectangle with boundary \( \partial R \) oriented in the counterclockwise direction relative to the currents running through it. If the coil has \( N \) turns around, then we have according to Ampère's Law

\[
\oint \mathbf{d} \mathbf{r} \times \mathbf{B}(\mathbf{r}) = N I_0 \mathbf{\hat{z}}
\]

On the side inside the cylinder contributions, but

\[
\partial R_1: \mathbf{d} \mathbf{r} = -L \mathbf{\hat{x}} \text{ with } I \in (0, L) \Rightarrow \mathbf{d} r \times \mathbf{B} = -dL \mathbf{\hat{x}}
\]

and thus

\[
\oint \mathbf{d} \mathbf{r} \times \mathbf{B}(\mathbf{r}) = - \int_0^L dL \mathbf{\hat{x}} \cdot B_x = -L B
\]

and thus

\[
B_x = -\frac{N I_0 \mathbf{\hat{z}}}{L}
\]
This means the field points to the left as we already could guess from the direction of the current elements with help of the right hand rule. The magnitude is

\[ |\mathbf{B}| = \frac{N I_{\text{loop}}}{L} \]

\( (a) \) Field of an infinitely long wire of finite cross section

Since

\[ \oint \mathbf{B} \cdot d\mathbf{S} = 0 \]

\( \mathbf{S} \)

For any closed surface \( S \), the field lines must be closed. From the cylindrical symmetry they must be circles around the wire. Thus, we make the ansatz

\[ \mathbf{B} = B(\rho) \hat{\phi} \]

where the axial cylindrical coordinates. We assume that the current density is uniform and pointing in position and direction:

\[ \mathbf{J} = \left\{ \begin{array}{ll}
\frac{i_z}{\pi a^2} & \text{if } -a < z < a \\
0 & \text{if } |z| > a
\end{array} \right. \]

\[ B = \frac{i_z}{\pi a^2} \]
Now we use Ampère's law with circles of radius \( s \) around the cylinder:

\[
\oint \mathbf{B} \cdot d\mathbf{S} = \int \mathbf{J} \cdot d\mathbf{V}
\]

Over line \( ds \) is parameterized as

\[
d\mathbf{r} = \mathbf{r}'(\theta) = s \mathbf{e}_3(\theta) = s [\cos \ \theta \mathbf{e}_x + \sin \ \theta \mathbf{e}_y]
\]

\[
d\mathbf{r}^2 = ds \mathbf{e}_3(\theta) = ds \mathbf{e}_3 (-\sin \ \theta \mathbf{e}_x + \cos \ \theta \mathbf{e}_y)
\]

For the surface integral we have

\[
d\mathbf{S} = \mathbf{S}' d\psi d\varphi \mathbf{e}_z \quad \text{with} \quad \mathbf{S}' \in (0, s)
\]

Ampère's law tells us

\[
\int \frac{d\mathbf{r}^2 \mathbf{B}(\mathbf{r})}{d\mathbf{n}} = \int_0^{2\pi} d\psi \cdot \mathbf{J}(\mathbf{r})
\]

So we bound

\[
\int \frac{d\mathbf{r}^2 \mathbf{B}(\mathbf{r})}{d\mathbf{n}} = \int_0^{2\pi} d\psi \int_0^s \mathbf{S}' \mathbf{e}_3(\theta) \cdot \frac{\mathbf{B}_\varphi(\mathbf{r}) \mathbf{e}_\varphi(\theta)}{d\mathbf{r}^2} \mathbf{B}(\mathbf{r})
\]

\[
= 2\pi s \mathbf{B}_\varphi(\mathbf{r}) \int_0^{2\pi} d\psi = 2\pi s \mathbf{B}_\varphi(\mathbf{r})
\]

and
Now we have to discuss the two cases:

\( \tilde{\mathbf{d}} \cdot \tilde{\mathbf{g}}(\mathbf{r}) = \int \int \int d\mathbf{s}' \int_0^{2\pi} d\phi' \int_0^a d\rho' \tilde{\mathbf{n}} (\mathbf{r}', \mathbf{r}) \cdot \mathbf{g}(\mathbf{r}) \)

\( \tilde{\mathbf{d}} \cdot \tilde{\mathbf{g}} = \int \int \int d\mathbf{s}' \int_0^{2\pi} d\phi' \int_0^a d\rho' \tilde{\mathbf{n}} (\mathbf{r}', \mathbf{r}) = \frac{i}{\pi a^2} \int_0^a d\rho' ds' = \frac{i}{\pi a^2} \int_0^a d\rho' ds' \cdot 2\pi = \frac{2i}{a^2} \frac{1}{2} \frac{\rho^2}{2} = \frac{i \rho^2}{a^2} \)

\( \Rightarrow 2\pi \rho B_q(\rho) = \frac{10i \rho^2}{a^2} \)

\( \Rightarrow B_q(\rho) = \frac{10i \rho}{2\pi a^2} \quad \text{for} \quad \rho < a \)

\( (b) \quad \rho > a \Rightarrow \tilde{\mathbf{d}} = \frac{i}{\pi a^2} \tilde{\mathbf{n}}(\mathbf{r}) \quad \text{for} \quad \rho > a \)

\( \Rightarrow \int \int \int d\mathbf{s}' \tilde{\mathbf{d}} = \frac{2i}{a^2} \int_0^a d\rho' ds' = i \)

\( \Rightarrow B_q(\rho) = \frac{10i \rho}{2\pi \rho} \quad \text{for} \quad \rho > a \)
11. Faraday’s Law of Induction

In this chapter we begin the investigation of time-dependent electromagnetic phenomena. We need some experimental input first, as usual in physics!

11.1 The Experiment by Faraday and Henry

Consider two loops as follows

See a better drawing on next page

We assume that the whole setup is given in this way for some time and we assume no currents are flowing in the circuits. If now we close the switch, $S_1$, in circuit 1. Then a current begins to flow. Clearly that’s a current dependent current, i.e.,

where $i_1(t)$ is the current in circuit 1.

Only after a long time, the total current will be

$$i_1(t) \to i_1(t \to \infty) = \frac{V}{R_1}$$

The current won’t jump because losses by $R_1$.

Now the remarkable finding by the physicist's Faraday and Henry has been that there is a current “induced”
The Faraday experiment

- Faraday’s Law of induction

\[ \oint_{\partial \Omega_2} \mathbf{d} \cdot \mathbf{E}(t, \mathbf{r}) = -\frac{d}{dt} \int_{\#2} \mathbf{d} \mathbf{S} \cdot \mathbf{B}(t, \mathbf{r}) \]  

(1)

- When closing the switch, in circuit #1 a **time-dependent** current starts to run

- this induces a **time-dependent** magnetic field \( \mathbf{B}_1 \), reaching through the loop in circuit #2

- there it induces an EMF, resulting in a current which induces a magnetic field \( \mathbf{B}_2 \) counteracting the built-up of \( \mathbf{B}_1 \) in the loop of circuit #2.
In circuit 2, although there is no battery underneath the loop, the loop must be an electromotive force (EMF) induced in circuit 2 along the resistor $R$ leading to a current $i_2(t)$.

$$i_2(t) = \frac{\mathcal{E}}{R} \quad (\mathcal{E} : \text{EMF along } R)$$

This current is measured with the ammeter $A_2$.

The current only varies during the instant $i_2(t)$ is changing.

This current only varies during the instant $i_2(t)$ is changing.

Faraday then concluded that the EMF must be due to the time changing $\mathbf{B}$ field produced by the free charges in circuit 1.

Faraday's Law of Induction

Faraday found out, presumably by experiment, that the electromotive force is given by the law:

$$\oint \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt} \int d\mathbf{S} \cdot \mathbf{B}$$

We look at the loop in circuit 1. Then we have a field $\mathbf{B}$ which is directed as indicated in circuit 2. This induces the following EMF:

$$\oint d\mathbf{r} \mathbf{E} \cdot d\mathbf{r} = +\int \frac{d}{dt} \left( \mathbf{E}_1 \cdot d\mathbf{r} \right)$$

$$\oint d\mathbf{r} \mathbf{E} \cdot d\mathbf{r} = \int \frac{d}{dt} \mathbf{E}_1 \cdot d\mathbf{r}$$

with a positive $\mathbf{E}_1$. But since $\frac{d}{dt} \mathbf{E}_1$ is positive at $t = 0$, we have the current $i_2(t)$ given by

$$\oint d\mathbf{r} \mathbf{E} \cdot d\mathbf{r} = \int \mathbf{E}_1 \cdot d\mathbf{r} = \int \frac{d}{dt} \mathbf{E}_1 \cdot d\mathbf{r}$$

with a positive $\mathbf{E}_1$.

The current $i_2(t)$ is

$$i_2 = \frac{d}{dt} \left( \mathbf{E}_1 \cdot d\mathbf{r} \right)$$

**Important:**

$\mathbf{E}$ and $\mathbf{B}$ are oriented relative to each other by the RHR

The direction of the induced field $\mathbf{B}$ is given by $\mathbf{E}$, as indicated by the RHR.
Thus O is positive and moving thus the direction is indicated. This gives a $B_2$ in loop 1 giving a induced current against the direction of the current. If the battery is removed O. That's known as

Lent's Law

If a current is induced by some charge, its direction is such that it opposes the change.

Note: It would be a disaster if the sign in Faraday's law could be lost. Then we would produce an in endless current. No longer have to worry about a loss of energy. Both loops would conspire to keep with the process. Thus current would flow from the battery to help with the process. This could violate the law of energy conservation since we would produce a nil amount of heat i the process out of an infinite amount of heat in the process.

Nothing. Thus could be in contra abnormal to experience.

It is important to note that due to

$$\phi d\mathbf{r} \cdot \mathbf{E} = -\frac{d}{dt} \int d\mathbf{S} \cdot \mathbf{B} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S}$$

$\phi d\mathbf{r}$ is in peril and a conserving field. Whenever the wave $\mathbf{E}$ is a peculiar sort a conservative field. Of course our laws be sketchy.

$\mathbf{E}$ is in peril and a conservative field. Of course our laws be sketchy.

Note also that $\mathbf{B}$ is not changeable independent of time. To differential reasons

1. $\mathbf{B}$ changes with time
2. The surface area $S$ changes with time
3. The direction of the surface normal relative to $\mathbf{B}$ changes
4. Any combination of this
\[ \frac{dEB}{dt} = -B \frac{dx}{dt} = -B \frac{dL}{dt} \]

Thus \( \frac{dL}{dt} \) is given by \( \frac{dL}{dt} = - \frac{B}{\mu} \frac{dx}{dt} \).
\( \Phi_B = \int \mathbf{B} \cdot d\mathbf{A} = \Phi_B \omega (ut) \)

\[ \int \mathbf{E} \cdot d\mathbf{A} = -\frac{d\Phi}{dt} = \mathbf{A} \times \mathbf{B} \omega \sin (ut) \]

\[ i(t) = \frac{\mathbf{A} \times \mathbf{B} \omega \sin (ut)}{r} \]

The force on the loop is such that it works against the motion which is in accordance with Lenz's Law!

II.4 Mutual, inductor, and self-oscillation

Coming back to our discussion of Faraday's experiment, we note that the current through a loop 0 produces a current in loop 2 which is due to the change in the magnetic flux \( \Phi_B \). In general, it is difficult to calculate the flux for the new geometry. But we know that \( \mathbf{B} \times \mathbf{A} \) and thus also \( \Phi_B \) is given:

\[ \Phi_B^{(2)} = \mathbf{A} \cdot \mathbf{B} \]

The sign has to be determined in each case by seeing the conventions for the orientation of the boundary area and the moment of the
In any case one often defines \( R > 0 \). Often one can also
obviously apply that's nice!

An example is self-induction. It gives the in induction of
an EMT in a coil itself as follows:

\[
\Phi_B = L i
\]

where \( L \) is called self-inductance of the coil. The units are

\[
[\Phi_B] = [B] = [\mathcal{F}] = \text{Tm}^2 = \frac{\text{Wb}}{\text{m}^2}
\]

\[
[T] = \frac{\text{V}}{\text{s}} = \text{H} \quad (\text{Henry})
\]

\[
\Rightarrow [L] = \frac{T}{\text{H}} = \frac{\text{V}s}{\text{Wb}}
\]

As an example we calculate the time dependence of the
current. We use Faraday's law to write the current as:

\[
\Phi_B = + Li > 0
\]

\[
\Rightarrow -L \frac{di}{dt} = \oint_S \mathbf{E} \cdot d\mathbf{s} = -V + iR
\]

\[
\frac{di}{dt} = -\frac{R}{L}i + \frac{1}{L}V
\]
To solve the equation, we note that if \( i_1 \) and \( i_2 \) are two solutions, we have:

\[
\frac{d(i_1 - i_2)}{dt} = -\frac{R}{L} (i_1 + i_2)
\]

This means we need a general solution \( c_1 \) for \( V = 0 \) and a special solution for \( V \neq 0 \). It's clear that the latter solution is given by:

\[
c_2 = \frac{V}{R}
\]

Thus, the general solution is

\[
0 = a \exp(-\frac{R}{L} t) + \frac{V}{R}
\]

If the switch is closed at \( t = 0 \), we have \( i(0) = 0 \) \( \Rightarrow a = -\frac{V}{R} \).

\[
i = \frac{V}{R} \left[ 1 - \exp\left(-\frac{R}{L} t\right) \right]
\]

\[\text{(108)}\]
11.5 Self-inductance of a solenoidal coil

A solenoided coil with a current \( i \) has the B field

\[ B = \frac{\mu_0 N i}{L} \]

(remember that derivation will rely on Ampère's law). The flux through 1 loop is

\[ \Phi_B = AB = \pi r^2 B \]

\[ \Phi_B = N \Phi_B = N \pi r^2 B = \mu_0 N^2 \pi \frac{r^2}{L} i \]

\[ \Rightarrow L = \frac{\mu_0 N^2 \pi \frac{r^2}{L}}{i} \quad \text{(solenoidal coil)} \]
Now we are ready to discuss circuits with the following elements:

- **Resistor**: Resistance \( R \) (Unit: Ohm = \( \Omega \))
- **Capacitor**: Capacitance \( C \) (Unit: Farad = \( F \))
- **Inductor**: Self-inductance \( L \) (Unit: Henry = \( H \))

Now we shall study some examples, how to find the equations for time-changing currents, voltages etc.

12.1 RLC Circuit connected to Battery

We always use Faraday's law to set the right signs!

Let's consider the simple circuit:

![Simple Circuit Diagram]

To find the right signs, we will differentiate simplifying the law to one loop only. We associate its self-inductance with it.

\[ \int 0 \, dB \]

We assign a current in an arbitrary direction. It is customary to use the direction as assumed, because of the
Fleming's law, a constant voltage. Thus we apply
Faraday's law. If we go counterclockwise along the arc:
\[ \oint d\vec{r} \cdot \vec{E} = -\frac{d}{dt} \int_\gamma d\vec{s} \cdot \vec{B} \]
Since we go against the current, we have
\[ \oint d\vec{r} \cdot \vec{E} = V - iR - \frac{dQ}{dt} \]
Since \( \vec{B} \) is such the loop goes into the plane, but \( d\vec{s} \) is
going out, we have
\[ \oint \vec{E} = -L \dot{i} \]
and thus
\[ V - iR - \frac{1}{C} \frac{dQ}{dt} = +L \frac{d\dot{Q}}{dt} \]
To use a different law equation, we use
\[ \dot{Q} = +\frac{dQ}{dt} \]
The sign is determined by the fact that \( Q \) on the \( + \) pole
of the capacitor, goes, \( Q \) on the \( - \) pole. Thus, we obtain
\[ L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + C Q = V \]
We shall discuss the solution of this equation in the next group because it is a very important case and it's covered only incompletely in this book.

### 12.2 The RC Circuit

![Diagram of an RC circuit with a capacitor, resistor, and voltage source.]

The inductor is self-inductance and Faraday's law gives

$$\oint \mathbf{E} \cdot d\mathbf{s} = 0$$

That's like the steady state. Then we have

$$\oint \mathbf{E} \cdot d\mathbf{s} = V - \frac{Q}{C}$$

Again, $i = \dot{Q} = \frac{dQ}{dt}$

$$\Rightarrow \quad R \dot{Q} + \frac{Q}{C} = V$$

This equation we can solve easily. Its general solution is the sum of the general solution of the homogeneous equation, and a particular solution of the non-homogeneous equation

$$R \dot{Q} + \frac{Q}{C} = 0$$

We make the ansatz

$$Q_+(t) = a \exp(2t)$$

with $a = \text{const.}$

$$\Rightarrow \quad R a \exp(2t) + \frac{1}{C} a \exp(2t) = 0$$

$$\Rightarrow \quad R a + \frac{1}{C} = 0$$

$$\Rightarrow \quad a = -\frac{1}{RC}$$
Thus
\[ Q_+ (t) = a \exp \left( -\frac{t}{RC} \right) \]

Particular solution of the inhom. eq.

12. \[ \dot{Q}_I + \frac{1}{C} Q_I = V \]
Since \( V = \text{const.} \), it is clear that the steady-state solution
\[ Q_I = \text{const.} \] The equation tells us of course the right value:
\[ Q_I = CV \]

General solution of the eq.

\[ Q(t) = Q_+(t) + Q_I(t) = a \exp \left( -\frac{t}{RC} \right) + CV \]

If we have \( Q = 0 \) at \( t = 0 \), we can solve for \( a \) as:
\[ Q(t) = CV \left[ 1 - \exp \left( -\frac{t}{RC} \right) \right] \]

\[ Q \]
\[ CV \]
\[ t \]

To find \( a \) we can just take the limit when
\[ \dot{Q}(t) = \frac{V}{R} \exp \left( -\frac{t}{RC} \right) \]
Thus the current jumps in self-induction by to the value \(\frac{V}{R}\) is an artifact of our neglect of
the self-inductance of the circuit. It's only a good approximation if the wires are not too long, because if
conduction velocities by which tells us that we cannot send signals with a speed larger than the speed of light.

12.3 The RL circuit

This example we have already treated in the previous chapter. Here
we wish that we get the voltage equation
by Faraday's law again as in the
example of the RLC circuit in
the beginning of this chapter

\[ \Phi = \mathbb{E} = V \]

\[ i(t) = R \frac{d\mathbb{E}}{dt} \]

We found the general solution to be (see p. 107–108)

\[ i(t) = \frac{V}{R} + a \exp\left(-\frac{R}{L} t\right) \]

If \( i(t=0) = 0 \) we can solve for \( a \) and find

\[ i(t) = \frac{V}{R} \left[ 1 - \exp\left(-\frac{R}{L} t\right) \right] \]
Now we can also calculate the case that we hook up some circuit to an AC voltage (as you get from a usual household plug)

\[ V(t) = V_0 \cos(\omega t) \]

**NB**
For a usual plug, we have

\[ V_0 = \sqrt{2 \cdot 120} V \approx 170 \text{ V} \]

and
\[ \omega = \frac{2\pi \cdot 60}{5} \approx 377 \frac{1}{5} \]

The RC circuit is treated as before

\[
\begin{array}{c}
\begin{array}{c}
\text{V} \\
\text{+}
\end{array} \\
\begin{array}{c}
\text{R} \\
\text{C}
\end{array}
\end{array}
\]

Ignoring the self-inductance with the above defined directions of \( i \) and \( dQ \), we have

\[
\dot{Q} \Rightarrow E = V - \frac{Q}{C} - IR = 0
\]

and
\[
i = \frac{dQ}{dt}
\]

So we have again

\[
RI + \frac{Q}{C} = V = V_0 \cos(\omega t)
\]

So we can take the solution of the homogeneous solution as before.
We look at the current now:

\[
i(t) = \frac{V}{R} \exp \left( -\frac{t}{RC} \right) \quad a = \text{const.}
\]
Inhomogeneous solution

Again, we think what should be the state after a long time when the homogeneous solution is damped (i.e. for times \( t \geq R/C \)): then the inductor has the current (and thus \( E = IR/C \)) and the capacitor has the same voltage as \( V \). However, \( Q \) is oscillating with the same frequency as \( V \). Suppose we set \( Q \) to be "in phase" with \( V \). So our ansatz is

\[
Q(t) = A \cos(wt) + B \sin(wt)
\]

with \( A, B = \text{constant} \). Then our ODE reads:

\[
R \frac{d}{dt} [A \cos(wt) + B \sin(wt)] + \frac{1}{C} \left[ A \omega_0 (\cos(wt) + B \sin(-wt)) \right]
\]

\[
= V_0 \cos(wt)
\]

This can be true for all \( t \) only if the coefficients of \( \cos(wt) \) and \( \sin(wt) \) on the same on both sides of the equation. So we find

\[
-R \omega_0 A + \frac{B}{C} = 0 \quad (1)
\]

\[
R \omega_0 B + \frac{A}{C} = V_0 \quad (2)
\]

(1) \( \Rightarrow \) \( A = \frac{-B}{RC} \)

This in (2) gives

\[
R \omega_0 B + \frac{B}{RC^2} = V_0
\]

\[
B \left[ \left( \frac{RC}{C^2} \right)^2 + 1 \right] = V_0
\]
\[ B = \frac{RwC^2}{1 + (RwC)^2} V_0 \]

\[ A = \frac{B}{RwC} = \frac{C}{1 + (RwC)^2} V_0 \]

So after a long time

\[ Q = \frac{C V_0}{1 + (RwC)^2} \left[ \cos(wt) + RwC \sin(wt) \right] \]

The current is

\[ i_{t \to \infty} = \frac{wC V_0}{1 + (RwC)^2} \left[ RwC \cos(wt) - \sin(wt) \right] \]

This we can also work differently, because we can use

\[ \omega \left( t + \beta \right) = \omega \left( t \right) \beta - 2 \sin \psi \sin \beta \]

Setting \( \psi = wt \) and \( \beta = \psi_0 \) we have

\[ \omega \left( wt + \psi_0 \right) = \omega \left( wt \right) \omega \psi_0 - \sin(wt) \sin \psi_0 \]

and we can work our solution as

\[ i_{t \to \infty} = \frac{wC V_0}{1 + (RwC)^2} \omega \left( wt + \psi_0 \right) \]

When \( \psi_0 \) is determined by

\[ \cos \psi_0 = \frac{RwC}{\sqrt{1 + (RwC)^2}} \quad \sin \psi_0 = -\frac{1}{\sqrt{1 + (RwC)^2}} \]
Note that such a \( \phi_0 \) exists because we have chosen the coefficient such that
\[
5\omega^2 \phi_0 + \omega^2 \phi_0 = 1
\]
as it must be.
Now
\[
\phi_0 = \tan^{-1} \frac{R \omega C}{\sqrt{1 - (R \omega C)^2}} \in \left[0, \frac{\pi}{2} \right]
\]
The sign is determined by the sign of \( \tan \phi_0 \) and the magnitude by \( \cos \phi_0 \). Thus, the \( \phi_0 \) is positive here.
We can write \( \phi_0 = -\omega T_0 \). Then we have
\[
i(t) = \frac{WC V_0}{\sqrt{1 + (R \omega C)^2}} \cos \left( \omega(t + T_0) \right)
\]
Since \( T_0 > 0 \) this means that the current is lagging a time to advance compared to \( V(t) \).

**Special case:** \( R = 0 \)

Then \( \omega L_0 = 0 \) and thus \( \phi_0 = +\frac{\pi}{2} \) and
\[
i(t) = WC V_0 \cos \left( \omega t + \frac{\pi}{2} \right)
= -WC V_0 \sin \left( \omega t \right)
\]
In this case the phase shift is \( \frac{\pi}{2} \) and thus the current "advances" \( V \) by exactly half of the period
\[
T = \frac{1}{\omega} = \frac{2\pi}{\omega}.
\]
The RL circuit with an AC voltage

We can now again use result from sub. 12.3:

\[ L \frac{di}{dt} + Ri = V = V_0 \cos (wt) \]

Also the homogeneous equation has the same solution:

\[ i(t) = A \exp \left( -\frac{R}{L} t \right) \quad a = \text{const}. \]

Only the inhomogeneous equation has to be solved again.

As in the previous example, a long time after switching on the voltage, we have

\[ i(t) \to A \cos (wt) + B \sin (wt) \]

\[ t \to \infty \]

\[ L \left( -A \cos (wt) + B \sin (wt) \right) + R \left[ A \cos (wt) + B \sin (wt) \right] = V_0 \cos (wt) \]

Comparing coefficients

\[ -WL A + RB = 0 \]

\[ WL B + RA = V_0 \]

Solving the linear equations has

\[ A = \frac{R V_0}{R^2 + \omega^2 L^2} \quad B = \frac{WL V_0}{R^2 + \omega^2 L^2} \]

\[ i(t) \to \infty = \frac{R V_0}{R^2 + \omega^2 L^2} \cos (wt) + \frac{WL V_0}{R^2 + \omega^2 L^2} \sin (wt) \]
In terms of a phase shift,

\[ i(t) = \text{max} \cos(\omega t + \phi_0) \]

\[ = \text{max} \left[ \cos(\omega t) \cos(\phi_0) - \sin(\omega t) \sin(\phi_0) \right] \]

\[ = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}} \left[ \frac{\omega}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t) + \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t) \right] \]

\[ \Rightarrow \cos(\phi_0) = \frac{R}{\sqrt{R^2 + \omega^2 L^2}} \quad \sin \phi_0 = \frac{-\omega L}{\sqrt{R^2 + \omega^2 L^2}} \]

Since \( \sin \phi_0 \geq 0 \) we have

\[ \phi_0 = -\arcsin \left( \frac{R}{\sqrt{R^2 + \omega^2 L^2}} \right) \in \left[ -\frac{\pi}{2}, 0 \right] \]

Some \( \phi_0 \) is possible when we have

\[ T_0 = -\frac{\phi_0}{\omega} > 0 \]

\[ i(t) = \text{max} \cos(\omega(t + T_0)) \]

The answer is behind the voltage. That's again due to Lenz's Law: the inductive current opposes the change of the current due to the change of voltage.

For \( R = 0 \) and \( \phi_0 = \frac{-\pi}{2} \) the answer is quite a phase behind the voltage.

For \( L = 0 \), \( \phi_0 = 0 \). Thus there is no phase shift.
12.6 Discussion of the RLC circuit with AC voltage

We shall discuss the RLC circuit in detail.

In the circuit shown on the right, we have the following equation

\[ \mathbf{E} = Ri + \frac{\mathbf{d}i}{\mathbf{c}} - \mathbf{V} = -L \frac{\mathbf{d}i}{\mathbf{dt}} \]

With the current flowing in the shown direction, we have

\[ i = \frac{dQ}{dt} \implies \frac{di}{dt} = \frac{d^2Q}{dt^2} \]

Then our differential equation reads

\[ L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = V(t) \]

The general solution is then as the sum of

a special solution of the equation itself. It does
not matter which solution you choose. We define
this solution as \( Q_H(t) \) (I = i homogeneous equation)

The general solution of the homogeneous equation
which we find by setting \( V(t) = 0 \):

\[ L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = 0 \]

\( H = \text{homogeneous equation} \)

\[ Q(t) = Q_H(t) + Q_I(t) \]
12.6.1. General solution for the homogeneous equation

\[ \frac{d^2 Q_H}{dt^2} + \beta \frac{d Q_H}{dt} + \frac{q_0 t}{c} = 0 \]

In the calculus course one shows that the most general solution is given by

\[ Q_H(t) = A Q_H^{(1)}(t) + B Q_H^{(2)}(t) \]

where \( Q_H^{(1)} \) and \( Q_H^{(2)} \) are arbitrary solutions, which must be linearly independent, and \( A, B \) are real constants.

They are determined by the initial conditions of \( Q_H(t) \):

Initial solution:

\[ Q(t=0) = Q_0, \quad \frac{d Q(t)}{dt} \big|_{t=0} = c_0 \]

To solve the homogeneous equation, we remember that the resistor dissipates energy into heat, and thus we expect the amplitude of the charge to "decay" over time. This we can model as:

\[ Q_H(t) = \exp(-\beta t) \Phi(t) \]

when \( \Phi(t) \) is a skill unknown function. The derivatives read:

\[ \frac{dQ_H}{dt} = \exp(-\beta t) \left[ \frac{d\Phi}{dt} - \beta \Phi \right] \]

\[ \frac{d^2 Q_H}{dt^2} = \exp(-\beta t) \left[ \frac{d^2 \Phi}{dt^2} - 2\beta \frac{d\Phi}{dt} + \beta^2 \Phi \right] \]
Plugging this in our homogeneous equation yields

\[ mx'' + (\alpha - 2L\beta) x' + (\frac{1}{2} - \beta + L\beta^2) x = 0 \]

Since \( mx'' = 0 \), we must have

\[ L \frac{d^2 q}{dt^2} + (\alpha - 2L\beta) \frac{dq}{dt} + (\frac{1}{2} - \beta + L\beta^2) q = 0 \]

This becomes simpler if we choose

\[ \alpha - 2L\beta = 0 \Rightarrow \beta = \frac{R}{2L} \]

Then our equation for \( q \) becomes

\[ \frac{d^2 q}{dt^2} = -\left( \frac{1}{LC} - \frac{R^2}{4L^2} \right) q \]

To solve this equation we have to distinguish several cases.

(a) \( \frac{1}{LC} - \frac{R^2}{4L^2} > 0 \) (oscillatory)

Then we can write

\[ \frac{d^2 q}{dt^2} = -w^2 q \] with \( w = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \).

This is the equation of the harmonic oscillator, and we know the general solution

\[ q(t) = A \cos (wt) + B \sin (wt) \]

since \( \frac{\sin (wt)}{w \cos (wt)} = \tan (wt) \) is not constant, we have
Indeed the most general solution!

In this case, we have the solution for our original equation:

\[
Q_H(t) = \exp\left(-\beta t\right) + C(t)
\]

\[
\Rightarrow Q_H(t) = \exp\left(-\beta t\right) \left[ A \cos(\omega t) + B \sin(\omega t) \right]
\]

with \( \beta = \frac{R}{2L} \) and \( \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \)

This shows that we have a damped oscillatory solution for the equation. The solution becomes very small for times

\[
t \gg \frac{1}{\beta} = \frac{2L}{R}
\]

and

\[
\left| \frac{1}{LC} - \frac{R^2}{4L^2} \right| = 0 \quad \text{(anomalous limit)}
\]

\[
\Rightarrow \frac{d^2q}{dt^2} = 0 \quad \Rightarrow \quad q(t) = at + b
\]

The original equation has this as the solution:

\[
q(t) = at + b
\]

with \( \beta = \frac{R}{2L} \)
(c) \( \frac{1}{LC} - \frac{R^2}{4L^2} < 0 \) (overdamping)

Thus we have

\[
\frac{d^2q}{dt^2} = q^2 + \text{will } \lambda = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \text{ e}^R
\]

The general solution for this is

\[ q(t) = A \cos(\lambda t) + B \sin(-\lambda t) \]

The solution to the original equation, thus is

\[
q(t) = A \cos(-\lambda t) + B \sin(-2\lambda t)
\]

with

\[
\begin{align*}
\lambda &= \frac{R}{2L} - \lambda^2 \\
\end{align*}
\]

Since \( \lambda < \frac{R}{2L} \) the solution is damped as it must be. The charge cannot grow indefinitely on the capacitor if there is no source.
12.6.2 Particular solution for the inhomogeneous equation

As we have seen, the solutions of the homogeneous equation are damped after a long time. With the DC voltage

\[ V(t) = V_0 \cos(\omega t) \]

after the homogeneous part is damped away we should have an oscillatory solution with the same frequency:

\[ q(t) = a \cos(\omega t) + b \sin(\omega t) \]

Plugging this into the equation

\[ L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = V_0 \cos(\omega t) \]

we find

\[ \frac{1}{C} \left[ a \left(1 - LC \omega^2\right) + b RC \omega_0 \right] \sin(\omega_0 t) + \]

\[ \frac{1}{C} \left[ b \left(1 - LC \omega^2\right) - a RC \omega_0 \right] \cos(\omega_0 t) = V_0 \cos(\omega_0 t) \]
It follows

\[ a \left( 1 - LC w_0^2 \right) + v \left( RC w_0 \right) = CV_0 \]
\[ a \left( 1 - LC w_0^2 \right) - a \left( RC w_0 \right) = 0 \]
\[ a \left( 1 - LC w_0^2 \right) = \left( RC w_0 \right) \]

The solution for this set of linear equations is

\[ a = \frac{C V_0 \left( 1 - LC w_0^2 \right)}{R C^2 w_0^2 + \left( 1 - LC w_0^2 \right)^2} \]
\[ V_0 \]
\[ \alpha = \frac{C^2 R W_0 V_0}{R C^2 w_0^2 + \left( 1 - LC w_0^2 \right)^2} \]

Now we work this out in terms of amplitude and phase shift

\[ Q = \hat{Q} \cos \left( \omega t + \phi_\alpha \right) \]
\[ a = \hat{Q} \cos \phi_a \]
\[ \alpha = \hat{a} \cos \phi_\alpha \]
\[ V_0 \]
\[ \sqrt{a^2 + \hat{a}^2} = \frac{V_0}{C} \]

\[ \alpha = \hat{Q} \cos \phi_\alpha \]

\[ \alpha = \frac{1 - LC w_0^2}{\sqrt{R C w_0^2} + (1 - LC w_0^2)^2} \]

\[ \cos \phi_\alpha = -\frac{1}{a} \]

So we have

\[ \phi_\alpha = -\arccos \left( \frac{1 - LC w_0^2}{\sqrt{R C w_0^2} + (1 - LC w_0^2)^2} \right) \in \left[ -\pi, 0 \right] \]

or

\[ \phi_\alpha = -\arccos \left( \frac{1}{\frac{w_0}{\sqrt{C} + \sqrt{L}} - \omega_0 L} \right) \in \left[ -\pi, \pi \right] \]
One other writes

\[ \hat{Q} = \frac{\hat{V}_0}{\omega_0 \sqrt{R^2 + \left(\frac{1}{\omega_0 C} - \omega_0 L\right)^2}} \]

Then one obtains for the current

\[ i(t) = \frac{d\hat{Q}}{dt} = -\frac{\hat{V}_0}{\sqrt{R^2 + \left(\frac{1}{\omega_0 C} - \omega_0 L\right)^2}} \sin(\omega_0 t + \phi_0) \]

\[ i(t) = \frac{\hat{V}_0}{\sqrt{R^2 + \left(\frac{1}{\omega_0 C} - \omega_0 L\right)^2}} \omega R (\omega_0 t + \phi_0 + \frac{\pi}{2}) \]

\[ i(t) = \hat{i} R (\omega_0 t + \phi_0) \]

with \( \phi_0 = \phi_0 + \frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \)

At a frequency \( \omega_0 = \omega R \) for which

\[ \frac{1}{\omega_0 C} - \omega_0 L = 0 \Rightarrow \omega R = \sqrt{\frac{1}{LC}} \]

The amplifier for the current becomes maximal. That's called a resonance.

\[ \hat{C} = \frac{\hat{V}_0}{R} \]

The phase shift between current and voltage then is

\[ \phi_0^{(r)} = \phi_0^{(r)} + \frac{\pi}{2} = 0 \], because \( \phi_0^{(r)} = -\frac{\pi}{2} \)
For \( \omega_0 = \omega_R \) the phase shift between current and voltage is 0, i.e., the current follows precisely the voltage. Current and voltage are "in phase".

For \( \omega_0 < \omega_R \) we have \( \phi_0 < 0 \). That means that the current lags behind the voltage. The formula shows that for low frequencies the self-inductance dominates the behavior. That's why \( \phi_0 < 0 \) as the capacitance dominates the behavior. For \( \omega = \omega_R \) we have \( \omega_0 > 0 \), i.e., the current lags behind the voltage. For \( \omega > \omega_R \), the self-inductance of the coil dominates and works against the changes of the current, thus slowing down the change of the voltage.
Average power of an AC circuit

As we have seen on several examples, we can describe the stationary state of an AC circuit with a source voltage with frequency \( f = \frac{\omega}{2\pi} \) by

\[
V(t) = V_0 \cos(\omega t)
\]

\[
i(t) = i_0 \cos(\omega t + \phi_0)
\]

The instantaneous power used by the circuit is

\[
P(t) = V(t) i(t) = V_0 i_0 \cos(\omega t) \left[ \cos(\omega t) \cos(\phi_0) - \sin(\omega t) \sin(\phi_0) \right]
\]

The average power is defined by

\[
\langle P \rangle = \frac{1}{T} \int_0^T dt \ P(t)
\]

where \( T = \frac{2\pi}{\omega} \) is the period.

Now,

\[
\cos(2\omega t) = \cos^2(\omega t) - \sin^2(\omega t)
\]

\[
= \cos^2(\omega t) - [1 - \cos^2(\omega t)]
\]

\[
= 2\cos^2(\omega t) - 1
\]

\[
\Rightarrow \quad \cos^2(\omega t) = \frac{1}{2} \left[ 1 + \cos(2\omega t) \right]
\]

\[
\sin(2\omega t) = 2 \sin(\omega t) \cos(\omega t)
\]

\[
\Rightarrow \quad \sin(\omega t) \cos(\omega t) = \frac{1}{2} \sin(2\omega t)
\]

\[
\Rightarrow \quad \int_0^T dt \ \cos^2(\omega t) = \int_0^T dt \ \frac{1}{2} \left[ 1 + \cos(2\omega t) \right]
\]

\[
= \frac{T}{2} + \frac{1}{4\omega} \sin(2\omega t) \bigg|_0^T = \frac{T}{2}
\]
$$\int_0^T dt \cos(\omega t) \cos(2\omega t) = \frac{1}{2} \int_0^T \sin(2\omega t)$$

$$= -\frac{1}{4\omega} \cos(2\omega t) \bigg|_0^T = 0$$

Thus,

$$\langle P \rangle = \frac{1}{2} V_0 \bar{I}_0 \cos \phi_0$$

Effective voltage and current for an AC circuit

The effective voltage and current of an AC source is defined by the real power it delivers to a DC source which delivers the same power at a purely Ohmian resistor.

From Faraday's law, neglecting L, we have

$$R \bar{I}(t) = V(t) = V_0 \cos(\omega t)$$

$$\implies \bar{I}(t) = \frac{V_0}{R} \cos(\omega t)$$

This means that $\bar{I}_0 = \frac{V_0}{R}$ and $\phi_0 = 0$. Thus the average power is

$$\langle P \rangle = \frac{1}{2} V_0 \bar{I}_0 = \frac{R^2 \bar{I}_0^2}{2} = \frac{V_0^2}{2R}$$

For a DC source we would have

$$P = R^2 \bar{I}_0^2 = \frac{V_0^2}{2R} \equiv \langle P \rangle$$

So:

$$\bar{I}_0^2 = \frac{1}{2} \bar{I}_0^2 \implies \bar{I}_0 = \frac{\bar{I}_0}{\sqrt{2}}$$

and

$$V_0^2 = \frac{1}{2} V_0^2 \implies V_0 = \frac{V_0}{\sqrt{2}}$$

These are the voltage and current for AC sources (like 110V for a typical household plug).