Online Quantum Field Theory Course
Exercises 1

The harmonic oscillator in the operator formalism

(a) Consider the simple harmonic oscillator in 1 dimension with the Hamiltonian

\[ H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2. \]  

(1)

Derive the eigenvectors and eigenvalues of the Hamiltonian, making use exclusively from the Heisenberg algebra

\[ [x, p] = i. \]  

(2)

Note that we set here and in the following \( \hbar = 1 \).

**Hint:** Use the annihilation and creation operators

\[ a = \sqrt{\frac{m\omega}{2}} x + i \sqrt{\frac{2\omega m}{\hbar}} p \]  

and \( a^\dagger \) to show that

\[ H = \left( a \dagger a + \frac{1}{2} \right) \omega \]  

(4)

and that the operator \( N = a \dagger a \) has the eigenvalues \{0, 1, 2, \ldots\} and that the corresponding eigenvectors \( |n\rangle \) are given recursively by \( a^\dagger |0\rangle = 0 \) and \( |n + 1\rangle = \frac{1}{\sqrt{n + 1}} a^\dagger |n\rangle \), where we have normalised the vectors to 1.

(b) From the results of exercise (a) calculate the ground state and the first two excited energy eigen states. For that use the known representation of quantum mechanics in position representation:

\[ x\psi(x) = x\psi(x), \quad p\psi(x) = \frac{1}{i} \frac{d}{dx} \psi(x). \]  

(5)

**Hint:** Use \( a\psi_0(x) = 0 \) to find the ground state wave function

\[ \psi_0(x) = N \exp\left(-\frac{m\omega}{2} x^2\right) \]  

(6)

and apply to it the recursion relation \( |n + 1\rangle = 1/\sqrt{n + 1} a^\dagger |n\rangle \).

(c)* The propagator of the harmonic oscillator (operator formalism)\(^1\)

Use the Heisenberg picture, where the full time dependence is at the selfadjoint operators, which represent observables. With the Hamiltonian (1) solve the operator equations of motion in the Heisenberg picture:

\[ \dot{x} = \frac{1}{i} [x, H], \quad \dot{p} = \frac{1}{i} [p, H] \]  

(7)

with the initial conditions

\[ x(t = 0) = x_0, \quad p(t = 0) = p_0. \]  

(8)

\(^1\)Exercises with an asterix are a little bit more difficult.
Now the generalised eigenvectors of the position operator is defined by
\[ x(t) |x,t\rangle = x |x,t\rangle. \] (9)

Show that then the propagator is given by
\[ U(x,t;x_0,0) = \langle x,t|x_0,0 \rangle. \] (10)

**Hint:** In the Heisenberg picture the state kets are time independent, and the wave function’s time dependence is completely given by the time dependence of the (generalised) eigenvectors: \( \psi(t,x) = \langle x,t|\psi \rangle. \) In this definition of the wave function insert a unit operator
\[ \mathbb{1} = \sum d \langle x_0,0| |x_0,0\rangle. \]

To find the explicit expression for \( U \) can be found by using the representation in terms of the generalised basis \( |x_0,0\rangle \), by solving the eigenvalue equation (9), using the identities
\[ x_0 \langle x_0,0|\psi \rangle = x_0 \langle x_0,0|\Psi \rangle, \quad p_0 \langle x_0,0|\psi \rangle = \frac{1}{i} \frac{d}{dx_0} \langle x_0,0|\Psi \rangle, \] (11)

where the solutions of the equations of motion (8-9) have to be used to express \( x(t) \) in terms of \( x_0 \) and \( p_0 \).

To find the normalisation of \( U \), use the fact that the time evolution has to be unitary at each \( t \), i.e.,
\[ \int dx_0 U(x,t;x_0,0)U^*(x',t;x_0,0) = \int dx_0 \langle x,t|x_0,0 \rangle \langle x_0,0|x',t \rangle = \langle x,t|x',t \rangle = \delta(x-x'). \] (12)

To determine finally the phase of \( U \) use the fact that
\[ U(x,t=0;x_0,0) = \langle x,0|x_0,0 \rangle = \delta(x-x_0). \] (13)

**Hint:** For the last step, calculate
\[ f(t,x) = \int dx_0 U(x,t;x_0,0) \exp(-ax_0^2) \] (14)
and take the limit \( \lim_{t\to0^+} f(t,x) = \exp(-ax^2) \) to determine \( U \)’s phase.

(d) How can one calculate from the propagator \( U(x,t;x_0,0) \) the thermodynamic partition sum
\[ Z(\beta) = \text{Tr} \exp(-\beta H)? \] (15)

**Hint:** Use the fact that from our Heisenberg picture calculation we know that
\[ U(x,t;x_0,0) = \langle x,t|x_0,0 \rangle = \langle \exp(i\beta H)x,0|x_0,0 \rangle = \langle x,0|\exp(-itH)|x_0,0 \rangle. \] (16)

Which value have you to chose for \( t \) to get \( \exp(-\beta H) \) on the right-hand side of (16)?

Then use the definition of the trace:
\[ \text{Tr} O = \int dx \langle x,0|O|x,0 \rangle. \] (17)

How can one express the thermodynamical partition sum as a path integral? What are the boundary conditions in this path integral?

(e) Use the complete set of energy eigenvectors, determined in (a), in the trace definition to obtain the partition sum:
\[ Z = \text{Tr} \exp(-\beta H) = \sum_{n=0}^{\infty} \langle n|\exp(-\beta H)|n \rangle. \] (18)

\(^2\)Note that the wave function and thus also the propagator are independent of the picture of time evolution!