Online Quantum Field Theory Course

Exercises 2

(1) Scalar fields

In the following, we look more carefully into the free field theory of a real scalar field coupled to an external current, defined by the Lagrangian

$$L_J = \frac{1}{2} \left( \partial_{\mu} \phi \right) \left( \partial^{\mu} \phi \right) - \frac{m^2}{2} \phi^2 + J \phi, \quad S_J[\phi] = \int d^4x L_J. \quad (1)$$

In these exercises we will use the path-integral to calculate the generating functional

$$Z_\eta[J] = \mathcal{N} \int D\phi \exp\{iS_J[\phi]\}. \quad (2)$$

We have introduced a small positive $\eta$ to regularise the path integral to project out the vacuum expectation value by setting

$$m^2 \to m^2_\eta = m^2 - i\eta. \quad (3)$$

Note that $\mathcal{N}$ is an indefinite factor which is unimportant for all practical purposes in vacuum quantum field theory (this changes at finite temperatures, but we shall come back to this later).

Show that (2) is correct (up to an indefinite contribution to $\mathcal{N}$) in the sense that we can derive it from the Hamiltonian path-integral formulation, which is the one we like to calculate in quantum field theory:

$$Z_\eta[J] = \mathcal{N}' \int D\phi \int DH \exp \left\{ i \int d^4x [\Pi \dot{\phi} - \mathcal{H}[\phi, \Pi] + J \phi] \right\}. \quad (4)$$

• (b) Start from (2) to calculate $Z_\eta[J]$ (up to an indefinite contribution to $\mathcal{N}$):

$$Z_\eta[J] = Z_\eta[0] \exp \left[ -\frac{i}{2} \int d^4x_1 d^4x_2 J(x_1) D_F(x_1 - x_2) J(x_2) \right]. \quad (5)$$

Hints: Substitute $\phi' = \phi - \varphi$, where $\varphi$ is the solution of the classical field equations at presence of the external current $J$:

$$\frac{\delta S_J[\phi]}{\delta \phi} \bigg|_{\phi=\varphi} = 0 \quad (6)$$

and write down the solution in terms of the Green’s function $D_F$ which is determined uniquely by our $\eta$ regulator. Show that

$$D_F(x) = \int \frac{d^4p}{(2\pi)^4} \frac{\exp(-ipx)}{p^2 - m^2 + i\eta}, \quad \varphi(x) = -\int D_F(x - x') j(x'). \quad (7)$$

(2)* The electromagnetic field

(a) Show that the Lagrangian

$$\mathcal{L}_J = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_{\mu} A^\mu$$

with $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$

describes the electromagnetic field, coupled to an external electric current $J_{\mu}$ by calculating the variation with respect to $A_{\mu}$ to derive the classical equations of motion.
(b) Show that the current must be conserved, $\partial_\mu J^\mu = 0$ when there should exist a solution to the classical equations of motion (which are the Maxwell equations, of course).

(c) Try to go to the Hamiltonian formalism and elaborate on the difficulties.

**Hint**: There is no easy way to do this. So just try it naively and see, how it fails. The solution to this problem is quite complicated and, fortunately not necessarily needed to do QED within the path-integral formalism.

(d) Write down the naive Lagrangian path-integral formula for $Z_\eta[J_\mu]$, where $\eta$ is introduced by

$$\mathcal{L}_{J,\eta} = \mathcal{L}_J - \frac{i\eta}{2} A_\mu A^\mu$$

(9)

to regulate the path integral (think about the change of sign compared to the scalar field). Try to do the same program as in Exercise (1) for the scalar field. Why is there trouble?

**Hint**: Remember gauge invariance from your classical electromagnetics lecture.

(e) Why is it easy to solve this problem in classical electromagnetics?