Gapless Hartree-Fock approximations for the linear $\sigma$ model

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with

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Outline

1. Φ-derivable (2PI) approximation schemes
2. Symmetry violations in 2PI approximations
3. Solution for the linear $\sigma$ model
4. Conclusions
Local and bilocal sources

- Generating functional for (disconnected) Green’s functions
  \[ Z[J, B] = N \int D\phi \exp \left[ iS[\phi] + i \{ J_1 \phi_1 \}_1 + \frac{i}{2} \{ B_{12} \phi_1 \phi_2 \}_{12} \right] \]

- Generating functional for connected Green’s functions
  \[ W[J, B] = -i \ln Z[J, B], \quad \frac{\delta W}{\delta J_1} = \phi_1, \quad \frac{\delta W}{\delta B_{12}} = \frac{1}{2} (G_{12} + \phi_1 \phi_2) \]

- Legendre transform: 2PI generating functional
  \[ \Gamma[\phi, G] = W[J, B] - \{ J_1 \phi_1 \}_1 - \frac{1}{2} \{ (\phi_1 \phi_2 + iG_{12})B_{12} \}_{12} \]

- Saddle point expansion of the path integral
  \[ \Gamma[\phi, G] = S[\phi] + \frac{i}{2} \text{Tr} \ln(\beta^2 G^{-1}) + \frac{1}{2} \{ D_{12}^{-1} (G_{12} - D_{12}) \}_{12} + \Phi[\phi, G] \]

with \[ D_{12}^{-1} = \frac{\delta^2 S[\phi]}{\delta \phi_1 \delta \phi_2} \]
Want to find $\varphi$ and $G$ at vanishing external sources $\Rightarrow$ Equations of motion:

$$\frac{\delta \Gamma}{\delta \varphi_1} = j_1 + \{B_{12} \varphi_2\}_2 = 0, \quad \frac{\delta \Gamma}{\delta G_{12}} = -\frac{i}{2} B_{12} = 0$$

Second equation:

$$D^{-1}_{12} - G^{-1}_{12} = 2i \frac{\delta \Phi}{\delta G_{12}} = \Sigma_{12}$$

$\Phi$ generates skeleton diagrams for self-energy

$\Phi$ must be two-particle irreducible (2PI)

Saddle-point expansion of the path integral: $\Phi$ diagrams $\geq 2$ loops
Simple $\phi^4$ model

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial_\mu \phi) - \frac{m}{2} \phi^2 - \frac{\lambda}{2} \phi^4, \quad S[\phi] = \{\mathcal{L}_1\}_1$$

The functional:

$$i\Gamma[\phi, G] = iS[\phi] + \bigcirc + \bigcirc + \bigcirc + \bigcirc + \bigcirc + \cdots$$

Field equation of motion:

$$i(\Box + m^2)\phi = \bigcirc + \bigcirc + \bigcirc + \bigcirc + \cdots$$
Simple $\phi^4$ model

\[ \mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial_\mu \phi) - \frac{m}{2} \phi^2 - \frac{\lambda}{2} \phi^4, \quad S[\phi] = \{ \mathcal{L}_1 \}_1 \]

The functional:

\[ i\Gamma[\phi, G] = iS[\phi] + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \quad \]

Self energy:

\[ -i\Sigma_{12} = \quad + \quad + \quad + \quad + \quad + \quad + \quad \]
Why should one use the $\Phi$ functional?

- Provides a self-consistent set of equations of motion
- Approximations yield equations, which
  - lead to conserved expectation values of Noether currents
  - $i\Gamma = \ln Z$ at the solution
    (a non-perturbative approximation of the partition sum)
  - allows consistent determination of thermodynamical and dynamical properties through analytic properties of Green's functions
- especially useful for description of particles and resonances with finite mass width
- only way to find self-consistent equation with these properties!

[Baym 1962]
Breaking of symmetries: The $O(N)-\sigma$ model

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \vec{\phi})(\partial^\mu \vec{\phi}) - \frac{m}{2} \vec{\phi}^2 - \frac{\lambda}{4N} \left( \vec{\phi}^2 \right)^2$$

- Action symmetric under global $O(N)$ rotations of $\vec{\phi}$
- Symmetry linear $\Rightarrow$ exact Quantum action also symmetric
- Perturbative loop expansion $=$ power expansion in $\hbar$ $\Rightarrow$ also symmetric at any finite order of pert. theory
- If symmetry spontaneously broken ($m^2 < 0$), from this symmetry alone follows Goldstone’s theorem: There are $N - 1$ massless Goldstone bosons
- Long known [Baym, Grinstein 1977]: $\Phi$-derivable approximations break the symmetry explicitly!
- Goldstone’s theorem also violated
Gapless $\Phi$-derivable approximations

- $\Phi$-derivable approximation which fulfills Nambu-Goldstone theorem
  
  [Yu. B. Ivanov, F. Riek, J. Knoll 2005]

  $\implies$ Construct “correction” $\Delta \Phi$ to $\Phi$ functional such that
  - Nambu-Goldstone theorem is fulfilled in spont. broken phase
  - in symmetric phase: same EoMs in symmetric phase as original approximation
  - EoM for mean field unchanged

- for Hartree-Fock approximation

  $\Phi_{gHF} = \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \Delta \Phi$

  $\Delta \Phi = -\frac{\lambda}{2N} \left[ N Q_{ab} Q_{ab} - (Q_{aa})^2 \right]$

  $Q_{ab} = \int_{\beta} d^4k G_{ab}(k)$
mass-independent renormalization scheme

\[ \Sigma_{\text{vac}}(\phi = 0, m^2 = \mu^2, p^2 = 0) = 0, \]
\[ \partial_m \Sigma_{\text{vac}}(\phi = 0, m^2 = \mu^2, p^2 = 0) = 0, \]
\[ \partial_p \Sigma_{\text{vac}}(\phi = 0, m^2 = \mu^2, p^2 = 0) = 0. \]

preserves \( O(N) \) symmetry

only vacuum counter terms needed in \( \Phi \)-derivable scheme [HvH, J. Knoll 2002]

similar conditions used for effective potential
Solutions for $O(4)$ model in chiral limit

- With $\mu = 600$ MeV
- fixed physical parameters in vacuum: $m_\sigma = 600$ MeV, $f_\pi = 93$ MeV

stable and meta-stable solutions
2nd-order phase transitions
$m_\pi = 0$ in spont. broken phase ($\phi = 0$)
Solutions for $O(4)$ model in chiral limit

- With $\mu = 600$ MeV
- fixed physical parameters in vacuum: $m_\sigma = 600$ MeV, $f_\pi = 93$ MeV

Another high-mass metastable branch

No solutions at $T > T_{\text{end}}$

Effective renormalized coupling becomes high!

Approximation unreliable
Conclusions

- For linear $O(N)$-$\sigma$ model
  - $\Phi$-derivable (2PI) gapless approximations
  - renormalizable with symmetry-preserving vacuum counter terms
  - renormalization-scale independent vacuum solutions
  - stability of vacuum model: $\mu > \mu_0$
  - at finite temperature: 2nd-order phase transition(s)
  - various stable and meta-stable solutions
  - model breaks down at $T > T_{\text{end}}$

- remaining problems
  - at finite $T$: renormalization-scale dependence
  - deviation of renormalization-group $\beta$ from perturbation theory at the same order [E. Braaten, E. Petitgirard 2005; C. Destri and A. Sartirana 2005]
  - reason: subtraction of “hidden divergence” of the coupling constant resummed only in one channel
  - only partial resummation $\Rightarrow$ breaking of crossing symmetry at orders higher than expansion parameter like $\lambda, \hbar$

- Feasibility of gapless $\Phi$-derivable approximations at higher orders including scattering (sunset diagrams)?