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Selfconsistent Conserved Approximation for $\pi$- and $\rho$-Mesons
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The Model

The $\rho$-mesons

- Renormalizable model for massive $\rho$-mesons $\Rightarrow$ Higgs-Kibble-formalism for Gauge theories
- Start with a $SU(2)$ duplett with gauged symmetry group

$$\mathcal{L}_1 = -\frac{1}{2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2}(D_{\mu}\Phi)^\dagger D^{\mu}\Phi - V(\Phi)$$

- Mexican hat potential $V(\Phi) = -\frac{\mu^2}{2}\Phi^\dagger\Phi + \frac{\lambda}{4}(\Phi^\dagger\Phi)^2$

- Physical gauge (around the stable vacuum):
  - $\rho$-fields become massive $m_{\rho}^2 = g^2\mu^2/(4\lambda)$
  - Three $\Phi$-degrees of freedom become $\rho$ degrees of freedom
  - One $\Phi$-degree of freedom gives a massive “Higgs-particle”
The Model

The Pions

- Introduce Pions as adjoint representation, i.e., SO(3)-triplet

\[ \mathcal{L}_2 = \frac{1}{2} (D_{\mu} \vec{\pi}) \cdot (D^{\mu} \vec{\pi}) - \frac{\lambda_2}{8} (\vec{\pi}^2)^2 - \frac{\lambda_3}{4} \pi^2 \Phi^\dagger \Phi \]

- Consistency condition:

\[ m_\pi^2 = \frac{2m_\rho^2}{g} \lambda_3 \]

Unitary Gauge - Physical Vertices I

- \( \pi \rho \)-interactions

- \( \rho \rho \)-interactions

- \( \pi \) self interactions
Remarks about Quantization

- Unitary gauge contains only physical dof. ⇒ manifestly unitary
- To get renormalizable gauge ⇒ Introducing $R_\xi$-gauges (‘t Hooft)
- $R_\xi$-gauge: manifestly renormalizable
- $R_\xi$-gauge: Faddeev-Popov-ghosts
- BRST-invariance ⇒ $S$-Matrix gauge invariant
- $R_\xi$-gauge has unitary gauge as limit ⇒ Renormalized theory also unitary
The Model

The Photon

► Extending the gauge group to $U(1) \times SU(2)$

► $U(1)$ unbroken ⇒ One of the four gauge bosons remains massless ⇒ photon

► Equations of Motion ⇒ Pions couple to photons only through $\rho$ ⇒ Vector-Meson-Dominance

The Form Factor

► Electromagnetic Form Factor of the Pion:

$$F(k^2) = \frac{\pi^- + \pi^+ - \pi^- - \pi^+}{e + e}$$

► Feynman rules: $\Gamma_{\rho\gamma} = i\delta^a M_\rho^2 e/g ⇒$

$$|F(s)|^2 = \frac{m_\rho^4}{[s - m_\rho^2 - \text{Re} \Pi_\rho(s)]^2 + [\text{Im} \Pi_\rho(s)]^2}$$
Fit of the parameters

Form factor and Phase Shift

- Using dimensional regularization and renormalization of the one-loop-self-energy diagrams

\[ -i\Sigma_\rho = \frac{\pi}{400} + \frac{\pi}{1000} \]

Fit of the parameters

**Total $\pi^+\pi^-$ elastic cross-section**

- **Four $\pi$-vertex**

\[ \Gamma^{abcd}(p_1, \ldots, p_4) = \begin{cases} A(s, t, u)\delta_{ab}\delta_{cd} + & \\
+ A(t, s, u)\delta_{ac}\delta_{bd} + & \\
+ A(u, t, s)\delta_{ad}\delta_{bc} & \end{cases} \]

- With the invariants \( s = (p_1 + p_3)^2, \ t = (p_1 - p_3)^2 \) and \( u = (p_1 - p_4)^4 \)

- **Feynman rules ⇒ invariant transition amplitude:**

\[ M_{fi}(s, t) = A(s, t, u) + A(t, s, u)|_{u=4m^2_\pi - s - t} \]

- **Total cross section:**

\[ \sigma_{\text{tot}} = \frac{1}{64\pi} \frac{1}{s(s - 4m_\pi^2)} \int_0^{4m_\pi^2} |M_{fi}(s, t)|^2 \]
Fit of the parameters

- With the parameters from the fitting to phase-shift and form-factor:

\[
\sigma_{\text{tot}} \end{equation}
\text{[mbarn]}
\sqrt{s} \text{[GeV]}

- The $\delta_1$-phase-shift

- Data from: Forgatt, Petersen, Nucl. Phys. B129 (1977) 89
Introduce a bilocal source term in addition to the local source term into the Z-functional:

\[ Z[J, K] = N \int D\phi \exp \left[ iS[\phi] + i \langle J_1 \phi_1 \rangle_1 + \frac{1}{2} \langle K_{12} \phi_1 \phi_2 \rangle_{12} \right] , \quad W = -i \ln Z \]

\[ \mathcal{C} = \mathcal{H}_- + \mathcal{H}_+ + \mathcal{M} \]

Functional Legendre transformation wrt. both \( J \) and \( K \):

\[ \Gamma[\phi, G] = W[J, K] - \langle \phi_1 J_1 \rangle_1 - \frac{1}{2} \langle (\phi_1 \phi_2 + iG_{12}) K_{12} \rangle_{12} \]

with \( \phi_1 = \frac{\delta W[J, K]}{\delta J_1} \) and

\[ G_{12} = -\frac{\delta^2 W[J, K]}{\delta J_1 \delta J_2} = -2i \left( \frac{\delta W[J, K]}{\delta K_{12}} - \frac{1}{2} \phi_1 \phi_2 \right) \]

Define \( \Phi = \Gamma_2 \) to be then 2PI vacuum diagrams with at least 1 loop:

\[ \Gamma[\phi, G] = S[\phi] + \frac{i}{2} \text{Tr} \ln(DG^{-1}) + \frac{i}{2} \langle G_{12}^{-1}(G_{12} - D_{12}) \rangle_{12} + \Phi[\phi, G] \]
Equations of motion: $J = K = 0$

$$\frac{\delta \Gamma[\varphi, G]}{\delta \varphi} = 0 \Leftrightarrow (\square - m^2)\varphi + \frac{\delta S_I}{\delta \varphi} + \frac{1}{2} \left\langle \frac{\delta \mathcal{D}^{-1}_{12}}{\delta \varphi} G_{12} \right\rangle_{12} + \frac{\delta \Phi[\varphi, G]}{\delta \varphi} = 0$$

$$\frac{\delta \Gamma[\varphi, G]}{\delta G} = 0 \Leftrightarrow -i\Sigma_{12} := -i(\mathcal{D}^{-1}_{12} - G^{-1}_{12}) = 2\frac{\delta (i\Phi[\varphi, G])}{\delta G}$$

Simple example: $\phi^4$-theory:

$$i\Phi = \begin{array}{c}
\includegraphics[scale=0.5]{phi4}\end{array} + \begin{array}{c}
\includegraphics[scale=0.5]{phi4_i}\end{array} + \begin{array}{c}
\includegraphics[scale=0.5]{phi4_ii}\end{array} + \cdots$$

with $\begin{array}{c}
\includegraphics[scale=0.5]{phi4_ii} = \varphi(x)
\end{array}$

$$x \quad = iG(x, y)$$

$$-ij(x) = \begin{array}{c}
\includegraphics[scale=0.5]{phi4_iii}\end{array} + \begin{array}{c}
\includegraphics[scale=0.5]{phi4_iv}\end{array} + \begin{array}{c}
\includegraphics[scale=0.5]{phi4_v}\end{array} + \cdots$$

$$-(\square + m^2)\varphi = j$$

$$-i\Sigma_{12} = \begin{array}{c}
\includegraphics[scale=0.5]{phi4_vi}\end{array} + \begin{array}{c}
\includegraphics[scale=0.5]{phi4_vii}\end{array} + \begin{array}{c}
\includegraphics[scale=0.5]{phi4_viii}\end{array} + \cdots$$
Selfconsistent approximations

Generating functional

- $\Phi[G, D]$: sum over all 2PI closed diagrams with at least two loops

\[
i\Phi[G, D] = \sum_{\text{all 2PI closed diagrams}} + \ldots
\]

- Variation with respect to Green’s functions $\Rightarrow$ self energies fulfilling Dyson’s equations

\[
\frac{\delta i\Phi}{\delta D} = -i\Pi_{\rho} = \\
\frac{\delta i\Phi}{\delta G} = -i\Sigma_{\pi} = 
\]

\[
D = D_0 + D_0 \Pi_{\rho} D \\
G = G_0 + G_0 \Sigma_{\pi} G
\]

- Sum up to a certain loop order $\Rightarrow$ Selfconsistent effective approximation

- Respects all conservation laws basing on global symmetries

- In thermal field theory: Thermodynamically consistent approximation
Renormalization

Can be seen as resummation of all self energy insertions ⇒ Infinities to all orders

Renormalizable theory ⇒ finite by renormalizing parameters already present in Lagrangian

Physical renormalization conditions

\[ \Sigma_\pi(m_\pi^2) = \partial_s \Sigma_\pi(m_\pi^2) = 0, \quad \Pi_\rho(0) = \partial_s \Pi_\rho(0) = 0 \]

Analytical properties of Green’s functions

\[ G(s) = \frac{1}{\pi} \int_0^\infty \! dm^2 \Delta(m^2, s) A(m^2) \text{ with } A(s) = -\text{Im} G(s) \]

\( \Delta(m^2, s) \): Feynman-propagator ⇒ integral kernels ⇒ can be renormalized using standard techniques

self consistent finite set of coupled integral equations solvable numerically by iteration

Tadpole in vacuum absorbed into mass renormalization
The $\pi$-Self-Energy

Results in vacuum

$\text{Re}\Sigma_\pi$ vs $s$ [GeV$^2$]

$\text{Im}\Sigma_\pi$ vs $s$ [GeV$^2$]

$-i\Sigma_\pi = \ldots$
Result in vacuum

The $\rho$-Self-Energy

$$\Re \Pi_\rho [\text{GeV}^2]$$

$$\Im \Pi_\rho [\text{GeV}^2]$$

$$-i\Pi_\rho = \text{circles at } s = s_0$$
Finite Temperature

Quantum Field Theory at finite Temperature

- Using the modified Schwinger Keldysh contour for equilibrium

\[ C = \mathcal{H}_- + \mathcal{H}_+ + \mathcal{M} \]

- Timeordering in vacuum → contour ordering

- Path integral formalism: Generating functional \( Z \) factorizes in real time and imaginary time part.

- Calculate \( \langle O \rangle = \text{Tr}[\exp(-\beta H)O] \)

- Wick’s theorem ⇒ Path ordered Green’s functions → Matrix Formalism

- Trace ⇒ (Anti-)Periodicity of fields → KMS condition
The 2-point Green’s functions can be expressed in terms of the spectral function:

\[ iG^{--}(p) = \int_0^\infty \frac{dk_0}{\pi} \frac{2i k_0 A(p_0, \vec{p})}{p_0^2 - k_0^2 + i\epsilon} + 2n(p_0) A(p^2, \vec{p}), \]

\[ iG^{++}(p) = -\int_0^\infty \frac{dk_0}{\pi} \frac{2i k_0 A(p_0, \vec{p})}{p_0^2 - k_0^2 - i\epsilon} + 2n(p_0) A(p^2, \vec{p}), \]

\[ iG^{+-}(p) = 2[\Theta(p_0) + n(p_0)] A(p_0, \vec{p}) = 2[1 + f(l_0)] \tilde{A}(p), \]

\[ iG^{-+}(p) = 2[\Theta(-p_0) + n(p_0)] A(p_0, \vec{p}) = 2f(l_0) \tilde{A}(p). \]

with

\[ \tilde{A}(p) = -\text{Im} G_R(p) = \text{sign} p_0 A(p) \]

and

\[ f(x) = \frac{1}{\exp(\beta x) - 1} n(x) = f(|x|) \]

Feynman rules for imaginary part:

\[ \text{Im} G_R = \frac{G^{+-} - G^{-+}}{2i} \quad (1) \]

Self energies and Dyson equation

\[ G_R(p) = \frac{1}{p^2 - m^2 - \Sigma_R}, \quad \text{Im} \Sigma_R = \frac{\Sigma^{--} - \Sigma^{+-}}{2i} \]

Causality \(\Rightarrow\) Kramers-Kronig-Relations (Dispersion relations):

\[ f(z) = \int_{-\infty}^{\infty} \frac{dz'}{\pi} \frac{\text{Im} f(z')}{(z' - z)(z' - z_r)^n} + \sum_{k=0}^{n-1} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k. \]

Crucial: Subtractions ONLY in vacuum parts of the self energies!
The selfconsistent equations

Breaking of Lorentz invariance due to temperature:

\[ \Pi_{\mu\nu} = -\Pi_T \Theta^T_{\mu\nu} - \Pi_L \Theta^L_{\mu\nu} \]
with:

\[ \Theta_{\mu\nu} = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, \]

\[ \Theta^T_{\mu\nu} = \begin{cases} 
0 & \text{if } \mu = 0 \text{ or } \nu = 0 \\
-\delta_{ij} + \frac{p_i p_j}{p^2} & \text{for } \mu, \nu \in \{1, 2, 3\}, \end{cases} \]

\[ \Theta^L_{\mu\nu} = \Theta_{\mu\nu} - \Theta^T_{\mu\nu} \]

Dyson equation for transverse gauge (Landau gauge):

\[ G^{\rho}_{\mu\nu} = -\frac{\Theta^L_{\mu\nu}}{p^2 - m^2 - \Pi_L} - \frac{\Theta^T_{\mu\nu}}{p^2 - m^2 - \Pi_T} \]

Calculate iteratively: Imaginary part of self energies (finite!):

\[ \text{Im } \Pi^L_{\mu\nu}(p) = -\frac{g^2}{2\pi^4} \int d^4l \left[ \frac{(l_0 \vec{p}^2 - l_0 \vec{p})^2}{\vec{p}^2 p^2} \right] \left[ f(l_0) - f(l_0 + p_0) \right] A^\pi(l + p) A^\pi(l) \]

\[ \text{Im } \Pi^T_{\mu\nu}(p) = -\frac{g^2}{4\pi^4} \int d^4l \left[ \frac{\vec{p}^2 - (\vec{l} \vec{p})^2}{\vec{p}^2} \right] \left[ f(l_0) - f(l_0 + p_0) \right] A^\pi(l + p) A^\pi(l) \]

\[ \text{Im } \Sigma(\pi) = -\frac{g^2}{\pi^4} \int d^4l \left[ f(l_0) - f(l_0 + p_0) \right] (2p_\mu + l_\mu)(2p_\nu + l_\nu) \times \]
\[ \times \left[ \Theta^\mu\nu_L(l) A^\rho_L(l) + \Theta^\mu\nu_T(l) A^\rho_T(l) \right] A^\pi(l + p) \]

Calculate real parts for the temperature part with a dispersion relation without subtractions.
Dilepton Rate

Kadanoff-Baym-Equations: Exact result for strong coupling:

\[ \frac{d^4 R}{d\sqrt{s}dP^3} \bigg|_{\vec{P}=0} = \frac{2\alpha^2}{(2\pi)^3} \frac{m^2_{\rho}}{g^2 s} A_{\rho}(\sqrt{s}, 0) f_B(\sqrt{s}) \]

Dilepton Production Rate

\[ \frac{d^4 R}{d\sqrt{s}d^3P} [\text{GeV}^{-3}] \]

\( T = 150\text{MeV}, 200\text{MeV} \)
Conclusions and Outlook

Conclusions

▶ Selfconsistent treatment of all particles with width is possible
▶ Medium modification of the widths and masses
▶ At $T > 0$: contributions below light cone $\rightarrow$ “Screening”

Work to do

▶ To avoid trouble with unphysical states in vector-particle propagators: Apply $\xi \gg 1$ (“Unitary Gauge”)
▶ Other way: Give up the gauge model and generate the $\rho$ from an effective pion model which does not contain unphysical states at all
▶ Exploit non-abelian part of the $\rho$-interaction
▶ Include other particles ($A_1$, $\omega$, $N$, $\Delta$, $\ldots$)
▶ Question of theoretical interest: Is it possible to extend the selfconsistent approximation scheme to a gauge invariant one?