Introduction to the dynamical models of heavy-ion collisions

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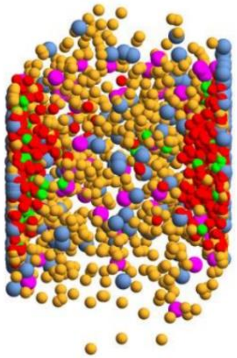
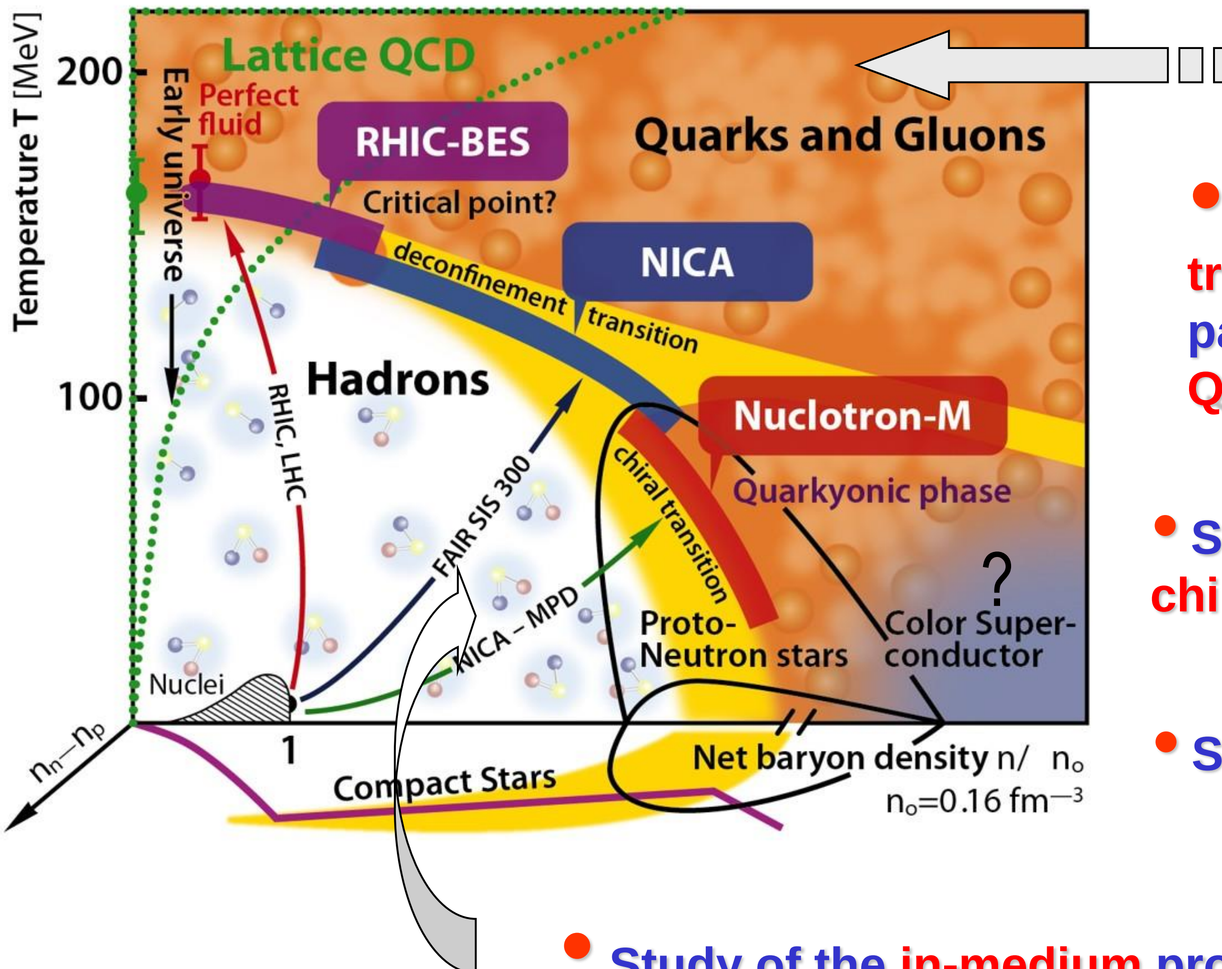


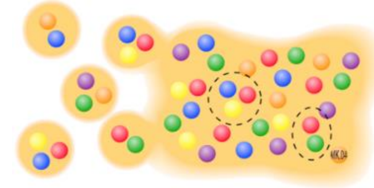
The CRC-TR211 Lecture Week “Lattice QCD and Dynamical models for Heavy Ion Physics”
Limburg, Germany, 10-13 December, 2019

The ‚holy grail‘ of heavy-ion physics:

The phase diagram of QCD



- Search for the critical point



- Study of the phase transition from hadronic to partonic matter – Quark-Gluon-Plasma
- Search for signatures of chiral symmetry restoration
- Search for the critical point
- Study of the in-medium properties of hadrons at high baryon density and temperature

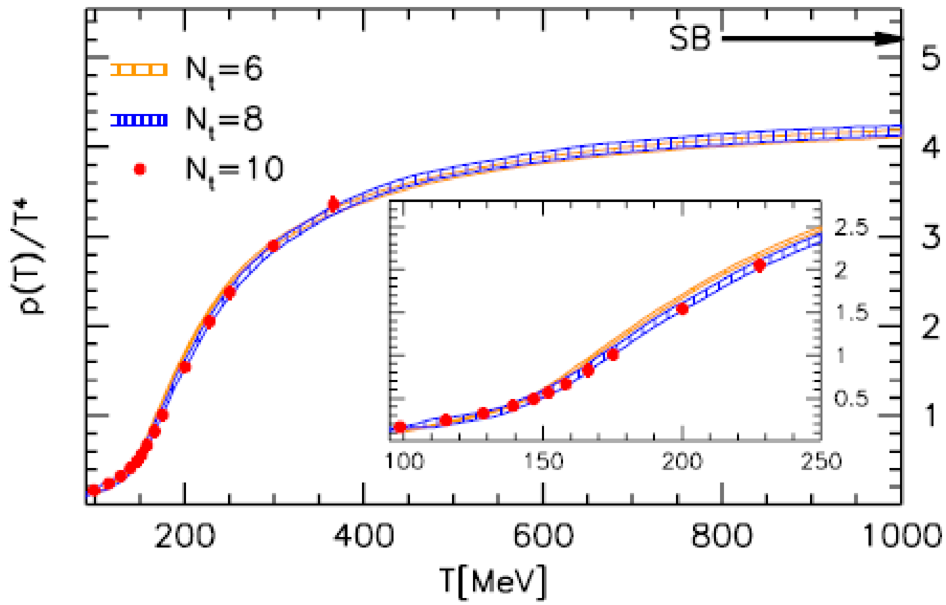
Theory: Information from lattice QCD

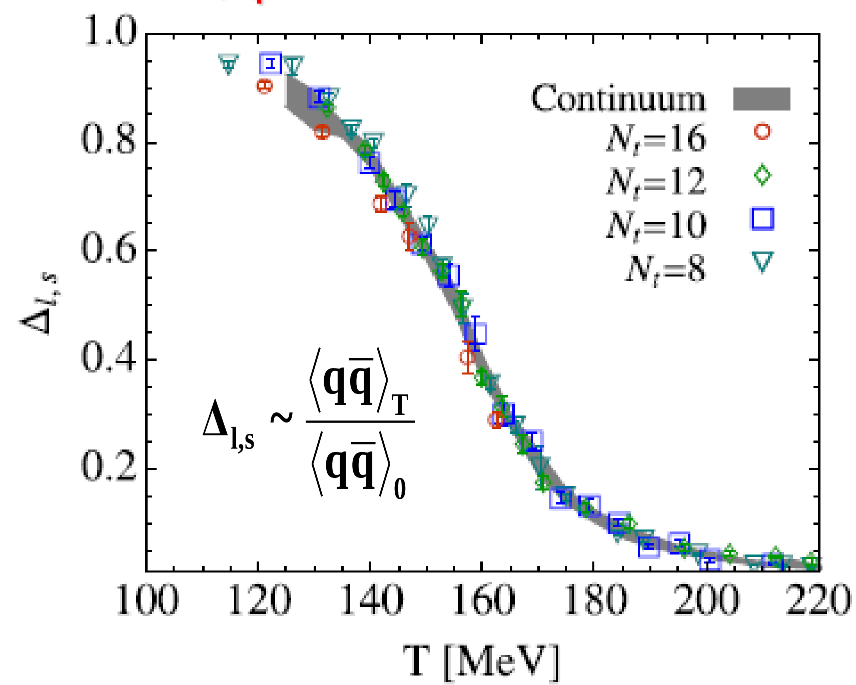
I. deconfinement phase transition **with increasing temperature**

+

II. chiral symmetry restoration **with increasing temperature**

lQCD BMW collaboration: $\mu_q=0$





- **Crossover: hadron gas → QGP**

- **Scalar quark condensate $\langle q\bar{q}\rangle$ is viewed as an order parameter for the restoration of chiral symmetry:**

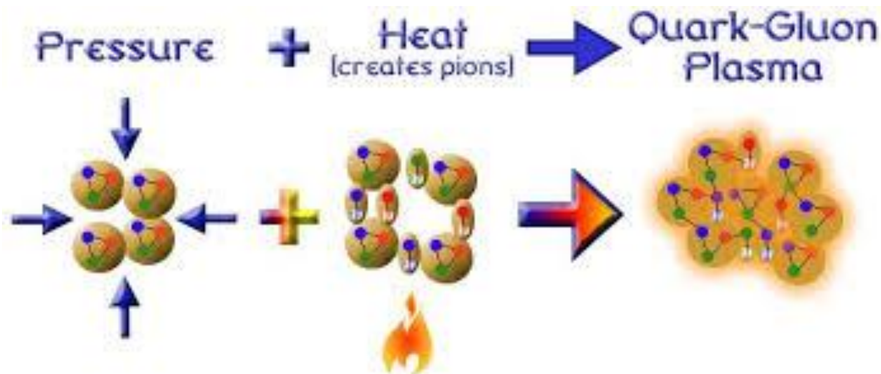
$$<\bar{q}q>=\begin{cases}\neq 0 & \text{chiral non-symmetric phase;}\\ =0 & \text{chiral symmetric phase.}\end{cases}$$

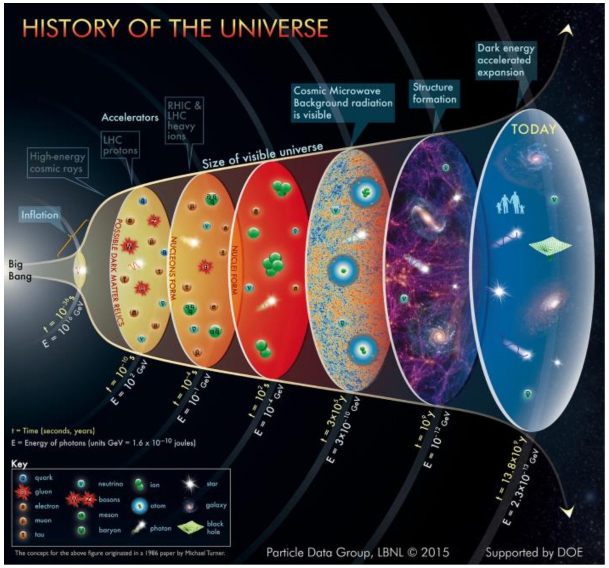
→ **both transitions occur at about the same temperature T_C for low chemical potentials**

Experiment: Heavy-ion collisions

❑ Heavy-ion collision experiment

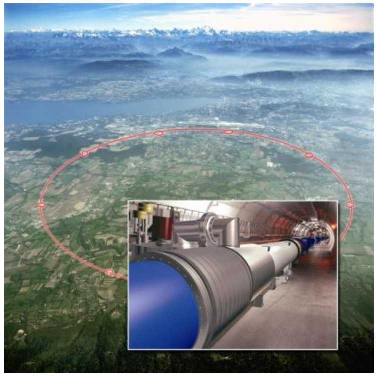
➔ ‚re-creation‘ of the Big Bang conditions in laboratory: matter at high pressure and temperature



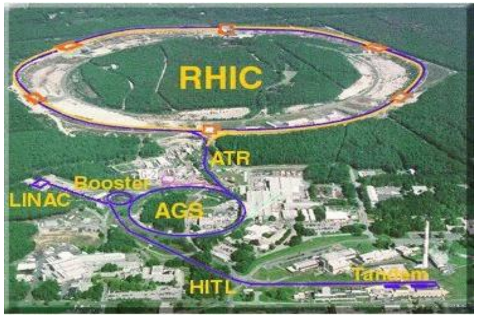


❑ Heavy-ion accelerators:

Large Hadron Collider - LHC (CERN):
Pb+Pb up to 574 A TeV

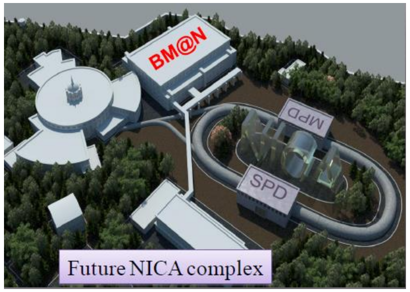


Relativistic-Heavy-Ion-Collider - RHIC (Brookhaven):
Au+Au up to 21.3 A TeV



Facility for Antiproton and Ion Research – FAIR (Darmstadt)
(Under construction)
Au+Au up to 10 (30) A GeV

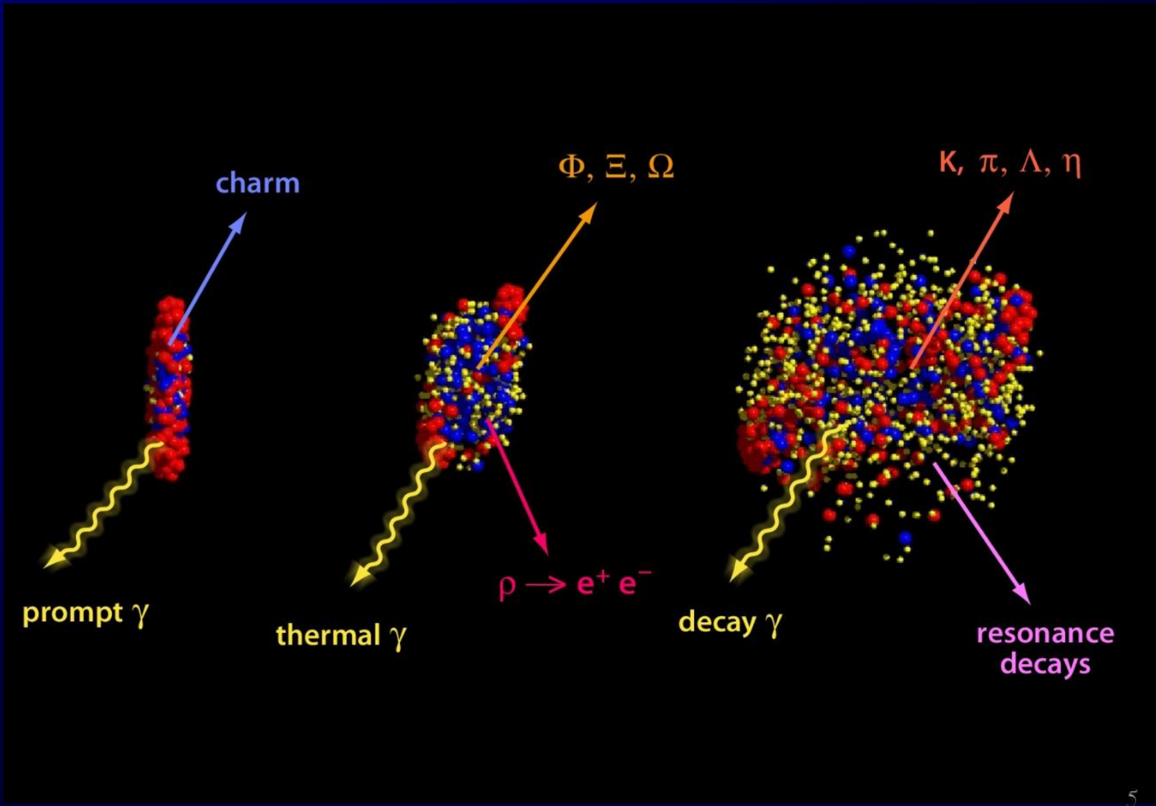
Nuclotron-based Ion Collider fAcility – NICA (Dubna)
(Under construction)
Au+Au up to 60 A GeV



Signals of the phase transition:

- Multi-strange particle enhancement in A+A
- Charm suppression
- Collective flow (v_1, v_2)
- Thermal dileptons
- Jet quenching and angular correlations
- High p_T suppression of hadrons
- Nonstatistical event by event fluctuations and correlations
- ...

Experiment: measures final hadrons and leptons



How to learn about physics from data?

Compare with theory!

Basic models for heavy-ion collisions

- **Statistical models:**

basic assumption: system is described by a (grand) canonical ensemble of non-interacting fermions and bosons in thermal and chemical equilibrium = thermal hadron gas at freeze-out with common T and μ_B

[- : no dynamical information]

- **Hydrodynamical models:**

basic assumption: conservation laws + equation of state (EoS); assumption of local thermal and chemical equilibrium
- Interactions are ‚hidden' in properties of the fluid described by transport coefficients (shear and bulk viscosity η, ζ, ..), which is 'input' for the hydro models

[- : simplified dynamics]

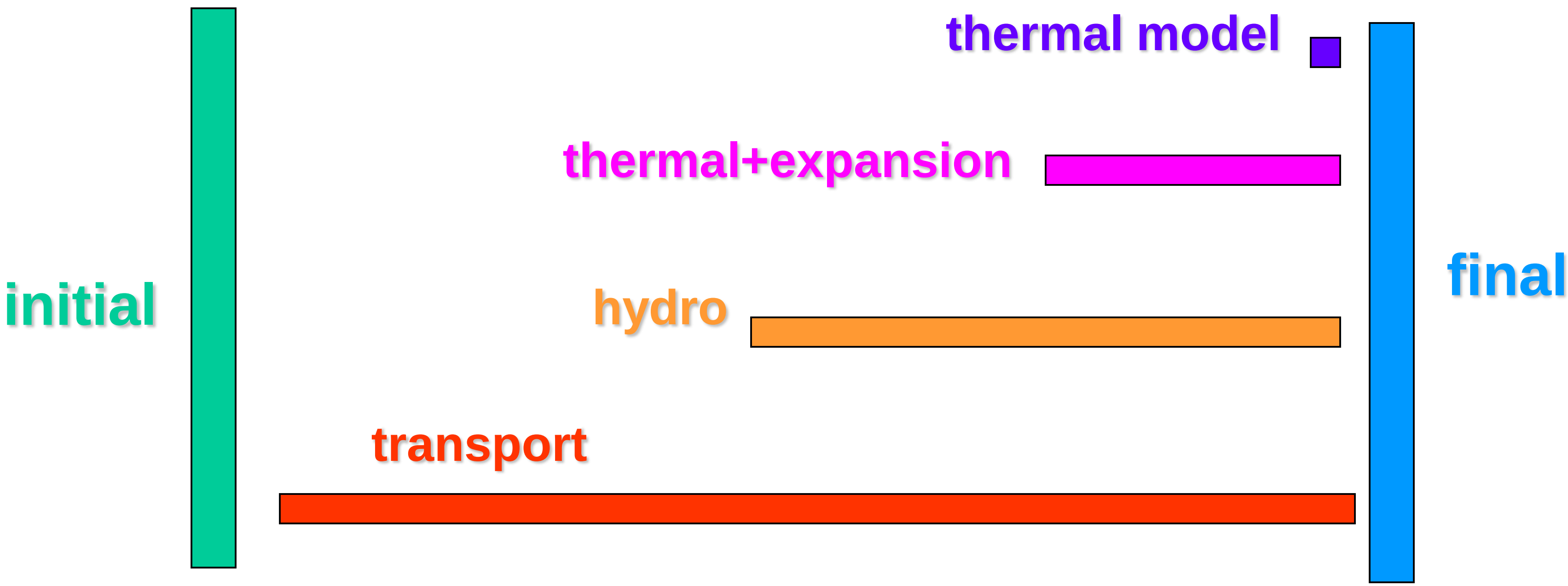
- **Microscopic transport models:**

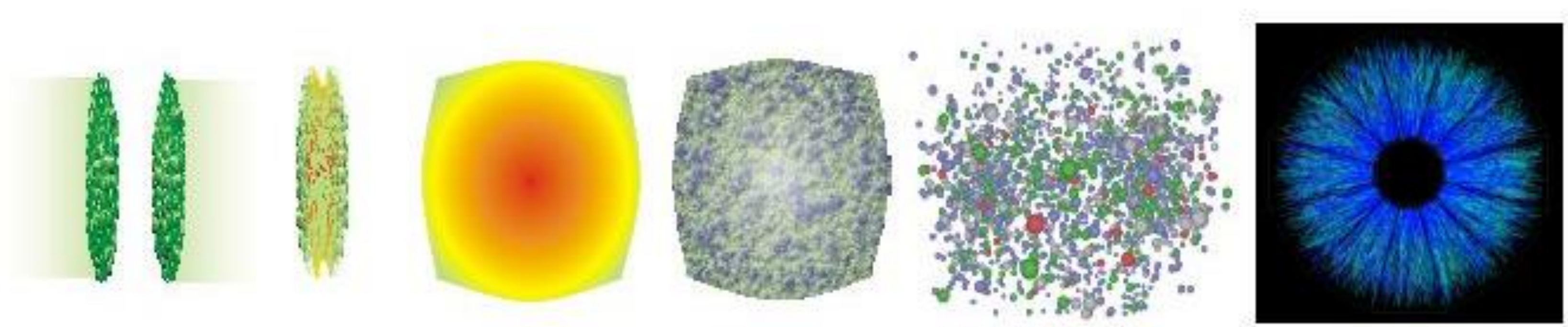
based on transport theory of relativistic quantum many-body systems
- Explicitly account for the interactions of all degrees of freedom (hadrons and partons) in terms of cross sections and potentials
- Provide a unique dynamical description of strongly interaction matter in- and out-off equilibrium:
- In-equilibrium: transport coefficients are calculated in a box – controled by lQCD
- Nonequilibrium dynamics – controled by HIC

Actual solutions: Monte Carlo simulations

[+ : full dynamics | - : very complicated]

Models of heavy-ion collisions





Dynamical description of heavy-ion collisions

The goal: to study the properties of strongly interacting matter under extreme conditions from a microscopic point of view

Realization: dynamical many-body transport approaches

This lecture:

1) Dynamical transport models (nonrelativistic formulation):
from the Schrödinger equation to Vlasov equation of motion ➔ BUU EoM

2) Density-matrix formalism: Correlation dynamics

3) Quantum field theory ➔ Kadanoff-Baym dynamics
➔ generalized off-shell transport equations

4) Transport models for HIC

1. From the Schrödinger equation to the Vlasov equation of motion

Quantum mechanical description of the many-body system

Dynamics of heavy-ion collisions is a many-body problem!

Schrödinger equation for the system of **N particles** in three dimensions:

$$i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, t) = H(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, t)\, \Psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, t)$$

nonrelativistic formulation

Hartree-Fock approximation:

- many-body wave function → $\Psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, t) = A \prod_{i=1}^{N} \psi_i(\vec{r}_i, t)$

  antisym. product of single-particle wave functions

- many-body Hamiltonian → single-particle Hartree-Fock Hamiltonian

$$H(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, t) = \sum_{i=1}^{N} T(\vec{r}_i) + V(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, t)$$

kinetic term N-body potential

$$T(\vec{r}) = -\frac{\hbar}{2m} \vec{\nabla}_r^2$$

$$\approx \sum_{i=1}^{N} T(\vec{r}_i) + \sum_{i<j}^{N} V_{ij}(|\vec{r}_i - \vec{r}_j|, t) \approx \sum_{i=1}^{N} h_i(\vec{r}_i, t)$$

kinetic term 2-body potential

Hartree-Fock equation

Time-dependent Hartree-Fock equation for a single particle *i*:

$$i\hbar \frac{\partial}{\partial t}\psi_i(\vec{r},t) = \hat{h}\psi_i(\vec{r},t)$$

Single-particle Hartree-Fock Hamiltonian operator: $\hat{h} = \hat{T} + \hat{U}_H - \hat{U}_F$

- **Hartree term:** $\hat{U}_H = \sum_{i(occ)} \int d^3r' \psi_i^*(\vec{r}',t) V(\vec{r}-\vec{r}') \psi_i(\vec{r}',t)$ $\quad \hat{T} = -\frac{\hbar}{2m}\vec{\nabla}_r^2$

  self-generated local mean-field potential

- **Fock term:** $\hat{U}_F = \sum_{i<N} \psi_i^*(\vec{r}',t) V(\vec{r},\vec{r}',t) \psi_i(\vec{r},t)$

  non-local mean-field exchange potential (quantum statistics)

➔ Equation-of-motion (EoM): propagation of particles in the self-generated mean-field:

$$i\hbar \frac{\partial}{\partial t}\psi_i(\vec{r},t) = \left(T(\vec{r},t) + U_H(\vec{r},t)\right)\psi_i(\vec{r},t) - \int d^3r' \; U_F(\vec{r},\vec{r}',t)\psi_i(\vec{r}',t)$$

local potential

We'll neglected the exchange (Fock) term

Note: TDHF approximation describes only the interactions of particles with the time-dependent mean-field !

In order to describe the collisions between the individual(!) particles, one has to go beyond the mean-field level ! (see Part 2: Correlation dynamics)

Single particle density matrix

- Introduce the single particle density matrix:

$$\rho(\vec{r},\vec{r}',t) \equiv \sum_{\beta_{occ}} \psi^*_\beta(\vec{r}',t)\psi_\beta(\vec{r},t)$$

Thus, the single-particle Hartree-Fock Hamiltonian operator can be written as

$$h(\vec{r},t) = T(\vec{r}) + \sum_{\beta_{occ}} \int d^3r' \, V(\vec{r}-\vec{r}',t)\rho(\vec{r}',\vec{r}',t) = T(\vec{r}) + U(\vec{r},t)$$

local potential

- Consider equation:

$$i\hbar \frac{\partial}{\partial t}\psi_\alpha(\vec{r},t) = h(\vec{r},t)\psi_\alpha(\vec{r},t) \qquad (1)$$

$\psi^*_\alpha(\vec{r}',t) * (1)$:

$$\psi^*_\alpha(\vec{r}',t) i\hbar \frac{\partial}{\partial t}\psi_\alpha(\vec{r},t) = \psi^*_\alpha(\vec{r}',t) h(\vec{r},t)\psi_\alpha(\vec{r},t) \qquad (2)$$

$(1)^+|_{for\ \vec{r}'} * \psi_\alpha(\vec{r},t)$:

$$-i\hbar\left[\frac{\partial}{\partial t}\psi^*_\alpha(\vec{r}',t)\right]\psi_\alpha(\vec{r},t) = h(\vec{r}',t)\psi^*_\alpha(\vec{r}',t)\psi_\alpha(\vec{r},t) \qquad (3)$$

→ $\sum_\alpha ((2)-(3))$:

Wigner transform of the density matrix

$$i\hbar\frac{\partial}{\partial t}\rho(\vec{r},\vec{r}\,',t)=\left[h(\vec{r},t)-h(\vec{r}\,',t)\right]\rho(\vec{r},\vec{r}\,',t) \quad (4)$$

$$\rho(\vec{r},\vec{r}\,',t)\equiv\sum_{\beta_{occ}}\psi^*_\beta(\vec{r}\,',t)\psi_\beta(\vec{r},t)$$

The single-particle Hartree-Fock Hamiltonian:

$$h(\vec{r},t)=T(\vec{r})+\int d^3r'\,V(\vec{r}-\vec{r}\,',t)\rho(\vec{r}\,',\vec{r}\,',t)$$

$$=T(\vec{r})+U(\vec{r},t)$$

kinetic term + potential (local) term

➔EoM:

$$\frac{\partial}{\partial t}\rho(\vec{r},\vec{r}\,',t)+\frac{i}{\hbar}\left[\frac{\hbar^2}{2m}\vec{\nabla}^2_r+U(\vec{r},t)-\frac{\hbar^2}{2m}\vec{\nabla}^2_{r'}-U(\vec{r}\,',t)\right]\rho(\vec{r},\vec{r}\,',t)=0 \quad (5)$$

Rewrite (5) using *x* instead of *r*

$$\frac{\partial}{\partial t}\rho(\vec{x},\vec{x}\,',t)+\frac{i}{\hbar}\left[\frac{\hbar^2}{2m}\vec{\nabla}^2_x+U(\vec{x},t)-\frac{\hbar^2}{2m}\vec{\nabla}^2_{x'}-U(\vec{x}\,',t)\right]\rho(\vec{x},\vec{x}\,',t)=0$$

Wigner transform of the density matrix

➔**EoM:**

$$\frac{\partial}{\partial t}\rho(\vec{x},\vec{x}',t)+\frac{i}{\hbar}\left[\frac{\hbar^2}{2m}\vec{\nabla}_x^2+U(\vec{x},t)-\frac{\hbar^2}{2m}\vec{\nabla}_{x'}^2-U(\vec{x}',t)\right]\rho(\vec{x},\vec{x}',t)=0$$

- **Instead of considering the density matrix ρ, let's find the equation of motion for its Fourier transform, i.e. the Wigner transform of the density matrix:**

$$f(\vec{r},\vec{p},t)=\int d^3s\ \exp\left(-\frac{i}{\hbar}\vec{p}\vec{s}\right)\rho\left(\vec{r}+\frac{\vec{s}}{2},\vec{r}-\frac{\vec{s}}{2},t\right)$$

$old:\vec{x}\qquad \vec{x}'$

New variables:

$$\vec{r}=\frac{\vec{x}+\vec{x}'}{2},\qquad \vec{s}=\vec{x}-\vec{x}'$$

$f(\vec{r},\vec{p},t)$ **is the single-particle phase-space distribution function**

Density in coordinate space: $\rho(\vec{r},t)=\frac{1}{(2\pi\hbar)^3}\int d^3p\ f(\vec{r},\vec{p},t)$

Density in momentum space: $g(\vec{p},t)=\int d^3r\ f(\vec{r},\vec{p},t)$

Wigner transformation + Taylor expansion

$$\frac{\partial}{\partial t}\rho(\vec{r}+\frac{\vec{s}}{2},\vec{r}-\frac{\vec{s}}{2},t) + \frac{i}{\hbar}\left[\frac{\hbar^2}{2m}\vec{\nabla}^2_{\vec{r}+\frac{\vec{s}}{2}} + U(\vec{r}+\frac{\vec{s}}{2},t) - \frac{\hbar^2}{2m}\vec{\nabla}^2_{\vec{r}-\frac{\vec{s}}{2}} - U(\vec{r}-\frac{\vec{s}}{2},t)\right]\rho(\vec{r}+\frac{\vec{s}}{2},\vec{r}-\frac{\vec{s}}{2},t) = 0 \qquad (1)$$

- Make Wigner transformation of eq.(2)

$$\int d^3s\ \exp\left(-\frac{i}{\hbar}\vec{p}\vec{s}\right)\frac{\partial}{\partial t}\rho\left(\vec{r}+\frac{\vec{s}}{2},\vec{r}-\frac{\vec{s}}{2},t\right) \qquad (2)$$

$$+\frac{i}{2m}\frac{\hbar^2}{\hbar}\int d^3s\ \exp\left(-\frac{i}{\hbar}\vec{p}\vec{s}\right)\left[\vec{\nabla}^2_{\vec{r}+\frac{\vec{s}}{2}} - \vec{\nabla}^2_{\vec{r}-\frac{\vec{s}}{2}}\right]\rho\left(\vec{r}+\frac{\vec{s}}{2},\vec{r}-\frac{\vec{s}}{2},t\right)$$

$$+\frac{i}{\hbar}\int d^3s\ \exp\left(-\frac{i}{\hbar}\vec{p}\vec{s}\right)\left[U(\vec{r}+\frac{\vec{s}}{2},t) - U(\vec{r}-\frac{\vec{s}}{2},t)\right]\rho\left(\vec{r}+\frac{\vec{s}}{2},\vec{r}-\frac{\vec{s}}{2},t\right) = 0$$

- Use that $\vec{\nabla}^2_{\vec{r}+\frac{\vec{s}}{2}} - \vec{\nabla}^2_{\vec{r}-\frac{\vec{s}}{2}} = 2\vec{\nabla}_{\vec{r}}\cdot\vec{\nabla}_{\vec{s}}$ (3)

- Consider $U(\vec{r}+\frac{\vec{s}}{2},t) - U(\vec{r}-\frac{\vec{s}}{2},t)$

Make Taylor expansion around r; $s \to 0$

$$(4) \qquad = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}\vec{s}\cdot\vec{\nabla}_{\vec{r}}\right)^n U|_{s=0} - \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{2}\vec{s}\cdot\vec{\nabla}_{\vec{r}}\right)^n U|_{s=0} = 2\sum_{odd}\left(\frac{1}{2}\vec{s}\cdot\vec{\nabla}_{\vec{r}}\right)^n U|_{s=0}$$

terms even in n cancel

$$\approx \vec{s}\cdot\vec{\nabla}_{\vec{r}}U(\vec{r},t)$$

Classical limit: keep only the first term $n=1$

Vlasov equation-of-motion

From (2) and (3),(4) obtain

$$\frac{\partial}{\partial t} f(\vec r,\vec p,t) + \frac{i}{2m}\frac{\hbar^2}{\hbar}\int d^3s\; \exp\left(-\frac{i}{\hbar}\vec p\vec s\right)\; 2\vec\nabla_{\vec r}\cdot\vec\nabla_{\vec s}\;\rho\left(\vec r+\frac{\vec s}{2},\vec r-\frac{\vec s}{2},t\right) \qquad (5)$$

$$+\frac{i}{\hbar}\int d^3s\; \exp\left(-\frac{i}{\hbar}\vec p\vec s\right)\; \vec s\cdot\vec\nabla_{\vec r}\, U(\vec r,t)\;\rho\left(\vec r+\frac{\vec s}{2},\vec r-\frac{\vec s}{2},t\right) = 0$$

(*): $f(\vec r,\vec p,t) = \int d^3s\; \exp\left(-\frac{i}{\hbar}\vec p\vec s\right)\;\rho\left(\vec r+\frac{\vec s}{2},\vec r-\frac{\vec s}{2},t\right)$

using (*)

Vlasov equation

- free propagation of particles in the self-generated HF mean-field potential:

$$\frac{\partial}{\partial t} f(\vec r,\vec p,t) + \frac{\vec p}{m}\vec\nabla_{\vec r}\; f(\vec r,\vec p,t) - \vec\nabla_{\vec r}U(\vec r,t)\;\vec\nabla_{\vec p}\, f(\vec r,\vec p,t) = 0 \qquad (6)$$

Eq.(6) is entirely classical (lowest order in *s* expansion).
Here *U* is a self-consistent potential associated with *f* phase-space distribution:

$$U(\vec r,t) = \frac{1}{(2\pi\hbar)^3}\int d^3r' d^3p\, V(\vec r-\vec r',t)\, f(\vec r',\vec p,t) \qquad (7)$$

Vlasov EoM

Vlasov equation of motion

- free propagation of particles in the self-generated HF mean-field potential:

$$\frac{\partial}{\partial t} f(\vec{r},\vec{p},t) + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}}\, f(\vec{r},\vec{p},t) - \vec{\nabla}_{\vec{r}} U(\vec{r},t)\, \vec{\nabla}_{\vec{p}} f(\vec{r},\vec{p},t) = 0$$

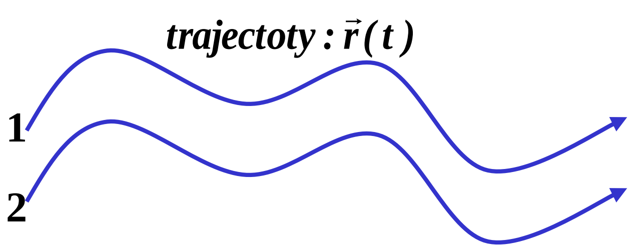
Vlasov EoM is equivalent to:

$$\frac{d}{dt} f(\vec{r},\vec{p},t) = 0 = \left[\frac{\partial}{\partial t} + \dot{\vec{r}}\vec{\nabla}_{\vec{r}} + \dot{\vec{p}}\vec{\nabla}_{\vec{p}}\right] f(\vec{r},\vec{p},t) = 0$$

➔ **Classical equations of motion :**

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{\vec{p}}{m}$$

$$\dot{\vec{p}} = \frac{d\vec{p}}{dt} = -\vec{\nabla}_{\vec{r}} U(\vec{r},t)$$

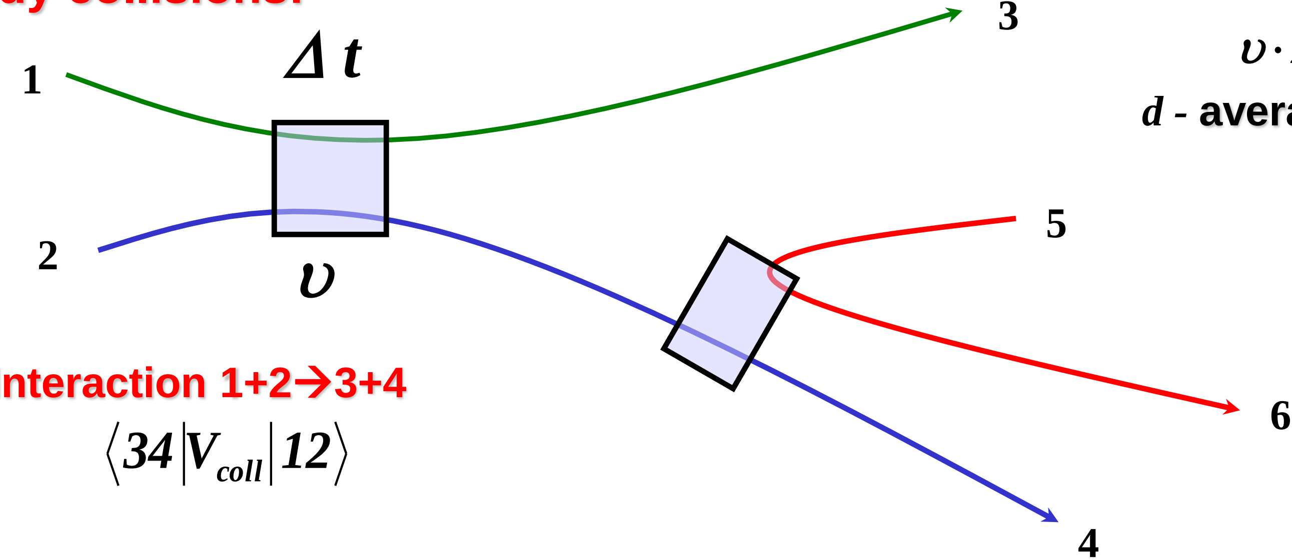


Note: the quantum physics plays a role in the initial conditions for *f*:
the initial *f* in case of fermions must respect the Pauli principle

Dynamical transport models with collisions

➔ In order to describe the collisions between the individual(!) particles, one has to go **beyond the mean-field level !** (See Part 2: Correlation dynamics)

add 2-body collisions:



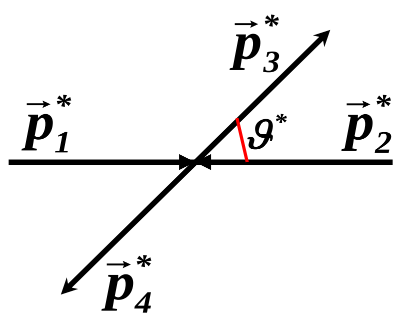
$$\upsilon \cdot \Delta t << \langle d \rangle$$

d - average distance

Interaction 1+2→3+4

$$\langle 34 | V_{coll} | 12 \rangle$$

In cms: $\vec{p}_1^{\,*} + \vec{p}_2^{\,*} = \vec{p}_3^{\,*} + \vec{p}_4^{\,*} = 0$



$$(\vec{r}_1, \vec{p}_1)\,(\vec{r}_2, \vec{p}_2) \rightarrow (\vec{r}_3, \vec{p}_3)\,(\vec{r}_4, \vec{p}_4)$$

- If the phase-space around $(\vec{r}_3, \vec{p}_3)$ and $(\vec{r}_4, \vec{p}_4)$ is essentially empty then the scattering is allowed,
- if the states are filled → Pauli suppression
  = Pauli principle

BUU (VUU) equation

Boltzmann (Vlasov)-Uehling-Uhlenbeck equation (NON-relativistic formulation!)
- free propagation of particles in the self-generated HF mean-field potential with an on-shell collision term:

$$\frac{d}{dt} f(\vec{r},\vec{p},t) \equiv \frac{\partial}{\partial t} f(\vec{r},\vec{p},t) + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}} \; f(\vec{r},\vec{p},t) - \vec{\nabla}_{\vec{r}} U(\vec{r},t)\,\vec{\nabla}_{\vec{p}} f(\vec{r},\vec{p},t) = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

Collision term for 1+2→3+4 (let's consider fermions) :

Probability including Pauli blocking of fermions

$$I_{coll} \equiv \left(\frac{\partial f}{\partial t}\right)_{coll} \Rightarrow \frac{1}{((2\pi)^3)^3}\int d^3p_2\, d^3p_3\, d^3p_4 \;\cdot w(1+2\rightarrow 3+4)\cdot P$$

$$\times (2\pi)^3\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\;(2\pi)\delta\left(\frac{\vec{p}_1}{2m_1}+\frac{\vec{p}_2}{2m_2}-\frac{\vec{p}_3}{2m_3}-\frac{\vec{p}_4}{2m_4}\right)$$

Transition probability for *1+2→3+4:* $w(1+2\rightarrow 3+4) \Rightarrow \upsilon_{12}\cdot\frac{d^3\sigma}{d^3q}$

where $\upsilon_{12} = \frac{\hbar}{m}\,|\,\vec{p}_1-\vec{p}_2\,|$ **- relative velocity of the colliding nucleons**

$\frac{d^3\sigma}{d^3q}$ **- differential cross section, *q* – momentum transfer** $\vec{q} = \vec{p}_1 - \vec{p}_3$

BUU: Collision term

$$I_{coll} = \frac{4}{(2\pi)^3} \int d^3p_2\, d^3p_3 \int d\Omega\ |v_{12}|\, \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \cdot \frac{d\sigma}{d\Omega}(1+2 \rightarrow 3+4) \cdot P$$

Probability including Pauli blocking of fermions:

$$P = f(\vec{r},\vec{p}_3,t)\, f(\vec{r},\vec{p}_4,t)\, [1 - f(\vec{r},\vec{p}_1,t)]\, [1 - f(\vec{r},\vec{p}_2,t)]$$
$$- f(\vec{r},\vec{p}_1,t)\, f(\vec{r},\vec{p}_2,t)\, [1 - f(\vec{r},\vec{p}_3,t)]\, [1 - f(\vec{r},\vec{p}_4,t)]$$
$$\equiv f_3 f_4 (1 - f_1)(1 - f_2) - f_1 f_2 (1 - f_3)(1 - f_4)$$

Gain term
3+4→1+2

Loss term
1+2→3+4

Pauli blocking factors for fermions *

For particle 1 and 2:
Collision term = Gain term – Loss term

$$I_{coll} = G - L$$

Note: for bosons – enhancement factor *1+f* (where *f<<1*); often one neglects bose enhancement for HIC, i.e. *1+f →1

Dynamical transport model: collision terms

- BUU eq. for different particles of type *i=1,…n*

$$Df_i \equiv \frac{d}{dt} f_i = I_{coll}\left[f_1, f_2,..., f_n\right] \qquad (20)$$

Drift term=Vlasov eq. collision term

$$i: \quad Baryons: \ p, n, \Delta(1232), N(1440), N(1535),..., \Lambda, \Sigma, \Sigma^*, \Xi, \Omega; \ \Lambda_C$$

$$Mesons: \ \pi, \eta, K, \bar{K}, \rho, \omega, K^*, \eta', \phi, a_1,..., D, \bar{D}, J/\Psi, \Psi',...$$

➔ coupled set of BUU equations for different particles of type *i=1,…n*

$$\begin{cases} Df_N = I_{coll}\left[f_N, f_\Delta, f_{N(1440)},..., f_\pi, f_\rho,...\right] \\ Df_\Delta = I_{coll}\left[f_N, f_\Delta, f_{N(1440)},..., f_\pi, f_\rho,...\right] \\ ... \\ Df_\pi = I_{coll}\left[f_N, f_\Delta, f_{N(1440)},..., f_\pi, f_\rho,...\right] \\ ... \end{cases}$$

E.g., Nucleon transport in N,π,Δ system : $Df_N = I_{coll}$

(only 1←→2, 2←→2 reactions indicated here)

$$
\begin{aligned}
\frac{\partial f_N}{\partial t} &+ \vec{v}\cdot\frac{\partial f_N}{\partial r} - \nabla_r U_N \cdot \frac{\partial f_N}{\partial p} = I_{NN\to NN} + I_{N\Delta\to N\Delta} + I_{N\pi\to N\pi} + I_{NN\to N\Delta} + I_{NN\to\Delta\Delta} + I_{N\Delta\to\Delta\Delta} + I_{N\Delta\to NN} + I_{N\pi\to\Delta} \\
&= \frac{\pi}{\hbar}\frac{g_1 g_2}{4}\frac{(2\pi)^2(\hbar c)^4}{\mu_{NN}c^2\cdot\mu_{NN}c^2}\int\frac{d^3p_2}{(2\pi\hbar)^3}\int\frac{d^3p_1'}{(2\pi\hbar)^3}\int\frac{d^3p_2'}{(2\pi\hbar)^3}\frac{d\sigma}{d\Omega}(p_1'-p_1)\cdot(2\pi\hbar)^3\delta^3(p_1+p_2-p_1'-p_2')\delta(\varepsilon_1+\varepsilon_2-\varepsilon_1'-\varepsilon_2')\,\cdot \\
&\quad\left[\frac{p_1'-p_2'}{p_1-p_2}f_N(p_1')f_N(p_2')(1-f_N(p_1))(1-f_N(p_2)) - \frac{p_1-p_2}{p_1'-p_2'}f_N(p_1)f_N(p_2)(1-f_N(p_1'))(1-f_N(p_2'))\right] \\
&+ \frac{\pi}{\hbar}\frac{g_1 g_2}{4}\frac{(2\pi)^2(\hbar c)^4}{\mu_{N\Delta}c^2\cdot\mu_{N\Delta}c^2}\int\frac{d^3p_2}{(2\pi\hbar)^3}\int\frac{d^3p_1'}{(2\pi\hbar)^3}\int\frac{d^3p_2'}{(2\pi\hbar)^3}\frac{d\sigma}{d\Omega}(p_1'-p_1)\cdot(2\pi\hbar)^3\delta^3(p_1+p_2-p_1'-p_2')\delta(\varepsilon_1+\varepsilon_2-\varepsilon_1'-\varepsilon_2')\,\cdot \\
&\quad\left[\frac{p_1'-p_2'}{p_1-p_2}f_N(p_1')f_\Delta(p_2')(1-f_N(p_1))(1-f_\Delta(p_2)) - \frac{p_1-p_2}{p_1'-p_2'}N(p_1)f_\Delta(p_2)(1-f_N(p_1'))(1-f_\Delta(p_2'))\right] \\
&+ \frac{\pi}{\hbar}\frac{g_1 g_2}{4}\frac{(2\pi)^2(\hbar c)^4}{\mu_{N\pi}c^2\cdot\mu_{N\pi}c^2}\int\frac{d^3p_2}{(2\pi\hbar)^3}\int\frac{d^3p_1'}{(2\pi\hbar)^3}\int\frac{d^3p_2'}{(2\pi\hbar)^3}\frac{d\sigma}{d\Omega}(p_1'-p_1)\cdot(2\pi\hbar)^3\delta^3(p_1+p_2-p_1'-p_2')\delta(\varepsilon_1+\varepsilon_2-\varepsilon_1'-\varepsilon_2')\,\cdot \\
&\quad\left[\frac{p_1'-p_2'}{p_1-p_2}f_N(p_1')f_\pi(p_2')(1-f_N(p_1))(1+f_\pi(p_2)) - \frac{p_1-p_2}{p_1'-p_2'}N(p_1)f_\pi(p_2)(1-f_N(p_1'))(1+f_\pi(p_2'))\right] \\
&+ \frac{\pi}{\hbar}\frac{g_1 g_2}{4}\frac{(2\pi)^2(\hbar c)^4}{\mu_{NN}c^2\cdot\mu_{N\Delta}c^2}\int\frac{d^3p_2}{(2\pi\hbar)^3}\int\frac{d^3p_1'}{(2\pi\hbar)^3}\int\frac{d^3p_2'}{(2\pi\hbar)^3}\frac{d\sigma}{d\Omega}(p_1'-p_1)\cdot(2\pi\hbar)^3\delta^3(p_1+p_2-p_1'-p_2')\delta(\varepsilon_1+\varepsilon_2-\varepsilon_1'-\varepsilon_2')\,\cdot \\
&\quad\left[\frac{p_1'-p_2'}{p_1-p_2}f_N(p_1')f_\Delta(p_2')(1-f_N(p_1))(1-f_N(p_2)) - \frac{p_1-p_2}{p_1'-p_2'}f_N(p_1)f_N(p_2)(1-f_N(p_1'))(1-f_\Delta(p_2'))\right] \\
&+ \frac{\pi}{\hbar}\frac{g_1 g_2}{4}\frac{(2\pi)^2(\hbar c)^4}{\mu_{NN}c^2\cdot\mu_{\Delta\Delta}c^2}\int\frac{d^3p_2}{(2\pi\hbar)^3}\int\frac{d^3p_1'}{(2\pi\hbar)^3}\int\frac{d^3p_2'}{(2\pi\hbar)^3}\frac{d\sigma}{d\Omega}(p_1'-p_1)\cdot(2\pi\hbar)^3\delta^3(p_1+p_2-p_1'-p_2')\delta(\varepsilon_1+\varepsilon_2-\varepsilon_1'-\varepsilon_2')\,\cdot \\
&\quad\left[\frac{p_1'-p_2'}{p_1-p_2}f_\Delta(p_1')f_\Delta(p_2')(1-f_N(p_1))(1-f_N(p_2)) - \frac{p_1-p_2}{p_1'-p_2'}N(p_1)f_N(p_2)(1-f_\Delta(p_1'))(1-f_\Delta(p_2'))\right] \\
&+ \frac{\pi}{\hbar}\frac{g_1 g_2}{4}\frac{(2\pi)^2(\hbar c)^4}{\mu_{N\Delta}c^2\cdot\mu_{\Delta\Delta}c^2}\int\frac{d^3p_2}{(2\pi\hbar)^3}\int\frac{d^3p_1'}{(2\pi\hbar)^3}\int\frac{d^3p_2'}{(2\pi\hbar)^3}\frac{d\sigma}{d\Omega}(p_1'-p_1)\cdot(2\pi\hbar)^3\delta^3(p_1+p_2-p_1'-p_2')\delta(\varepsilon_1+\varepsilon_2-\varepsilon_1'-\varepsilon_2')\,\cdot \\
&\quad\left[\frac{p_1'-p_2'}{p_1-p_2}f_\Delta(p_1')f_\Delta(p_2')(1-f_N(p_1))(1-f_\Delta(p_2)) - \frac{p_1-p_2}{p_1'-p_2'}N(p_1)f_\Delta(p_2)(1-f_\Delta(p_1'))(1-f_\Delta(p_2'))\right] \\
&+ \frac{\pi}{\hbar}\frac{g_1 g_2}{4}\frac{(2\pi)^2(\hbar c)^4}{\mu_{N\Delta}c^2\cdot\mu_{NN}c^2}\int\frac{d^3p_2}{(2\pi\hbar)^3}\int\frac{d^3p_1'}{(2\pi\hbar)^3}\int\frac{d^3p_2'}{(2\pi\hbar)^3}\frac{d\sigma}{d\Omega}(p_1'-p_1)\cdot(2\pi\hbar)^3\delta^3(p_1+p_2-p_1'-p_2')\delta(\varepsilon_1+\varepsilon_2-\varepsilon_1'-\varepsilon_2')\,\cdot \\
&\quad\left[\frac{p_1'-p_2'}{p_1-p_2}f_N(p_1')f_N(p_2')(1-f_N(p_1))(1-f_\Delta(p_2)) - \frac{p_1-p_2}{p_1'-p_2'}N(p_1)f_\Delta(p_2)(1-f_N(p_1'))(1-f_N(p_2'))\right] \\
&+ \int\frac{d^3p_\pi}{(2\pi\hbar)^3}\int\frac{d^3p_\Delta}{(2\pi\hbar)^3}|\langle p_\Delta|T|p_N p_\pi\rangle|^2\cdot(2\pi\hbar)^3\delta^3(p_N+p_\pi-p_\Delta)\delta(\varepsilon_N+\varepsilon_\pi-\varepsilon_\Delta)\,\cdot \\
&\quad[f_\Delta(p_\Delta)(1+f_\pi(p_\pi))(1-f_N(p_N)) - f_N(p_N)f_\pi(p_\pi)(1-f_\Delta(p_\Delta))]
\end{aligned}
$$

Full collision term consists of >10000 different particle combinations

- ➔ set of transport equations coupled via I_{coll} and mean field

Dynamical transport model: collision terms

Collision terms for *(N,Δ,π)* system: $\Delta \leftrightarrow \pi N$

** Relativistic formulation*

Eq. for Δ

$$Df_\Delta = \sum_{\pi,N} \frac{g}{(2\pi)^3} \int \frac{d^3 p_\pi}{E_\pi} \frac{d^3 p_N}{E_N} |M_{\Delta\leftrightarrow\pi N}|^2 \cdot \delta^4(p_\pi + p_N - p_\Delta) \times f_\pi(p_\pi)\, f_N(p_N)(1 - f_\Delta(p_\Delta))$$

$$- \sum_{\pi,N} \frac{g}{(2\pi)^3} \int \frac{d^3 p_\pi}{E_\pi} \frac{d^3 p_N}{E_N} |M_{\Delta\leftrightarrow\pi N}|^2 \cdot \delta^4(p_\pi + p_N - p_\Delta) \times f_\Delta(p_\Delta)\,(1 - f_N(p_N))(1 + f_\pi(p_\pi))$$

$$= \text{Gain } (\pi N \to \Delta) - \text{Loss } (\Delta \to \pi N)$$

Δ production *Δ decay*

Eq. for π

$$Df_\pi = \sum_{N,\Delta} \frac{g}{(2\pi)^3} \int \frac{d^3 p_\Delta}{E_\Delta} \frac{d^3 p_N}{E_N} |M_{\Delta\leftrightarrow\pi N}|^2 \cdot \delta^4(p_\pi + p_N - p_\Delta) \times f_\Delta(p_\Delta)\,(1 + f_\pi(p_\pi))(1 - f_N(p_N))$$

$$- \sum_{\pi,N} \frac{g}{(2\pi)^3} \int \frac{d^3 p_\Delta}{E_\Delta} \frac{d^3 p_N}{E_N} |M_{\Delta\leftrightarrow\pi N}|^2 \cdot \delta^4(p_\pi + p_N - p_\Delta) \times f_\pi(p_\pi)\, f_N(p_N)(1 - f_\Delta(p_\Delta))$$

$$= \text{Gain } (\Delta \to \pi N) - \text{Loss } (\pi N \to \Delta)$$

π production *π absorbtion*

by Δ decay *by nucleon*

Dynamical transport model: possible interactions

Consider possible interactions for the sytem of *(N,R,m)*,
where *N*-nucleons, *R*- resonances, *m*-mesons

- **elastic collisions:**

Baryon-baryon (BB):

$$NN \to NN$$
$$NR \to NR$$
$$RR' \to RR'$$

meson-Baryon (mB)

$$mN \to mN$$
$$mR \to mR$$

meson-meson (mm)

$$m\ m' \to m\ m'$$

Detailed balance:

$$a+b \leftrightarrow c$$
$$a+b \leftrightarrow c+d$$

- **inelastic collisions:**

Baryon-baryon (BB):

$$NN \leftrightarrow NR$$
$$NR \leftrightarrow NR'$$
$$NN \leftrightarrow RR'$$
...
$$BB \to X$$

meson-Baryon (mB)

$$mN \leftrightarrow R$$
$$mR \leftrightarrow R'$$
$$mB \leftrightarrow m'B'$$
...
$$mB \to X$$

meson-meson (mm)

$$m\ m' \leftrightarrow \tilde{m}$$
$$m\ m' \leftrightarrow m''m'''$$
...
$$m\ m' \to X$$

***X* - multi-particle state**

Elementary hadronic interactions

Consider all possible interactions – elastic and inelastic collisions - for the sytem of *(N,R,m)*, where *N*-nucleons, *R*- resonances, *m*-mesons, and resonance decays

Low energy collisions:

- **binary 2←→2 and 2←→3(4) reactions**
- **1←→2 : formation and decay of baryonic and mesonic resonances**

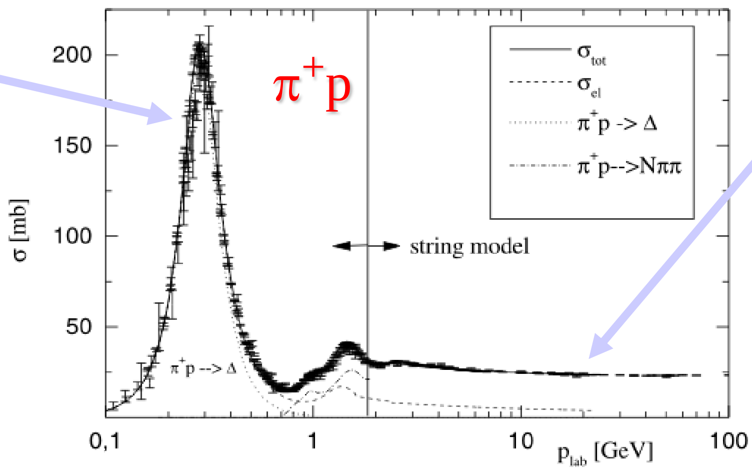
BB ←→ B´B´
BB ←→ B´B´m
mB ←→ m´B´
mB ←→ B´
mm ←→ m´m´
mm ←→ m´ . . .

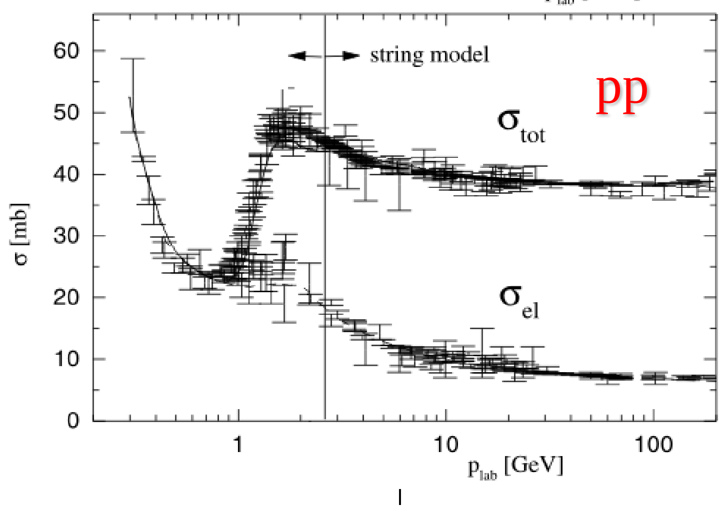
Baryons:

B = p, n, Δ(1232), N(1440), N(1535), ...

Mesons:

M = π, η, ρ, ω, ϕ, ...





High energy collisions:

(above $s^{1/2}$~2.5 GeV)

Inclusive particle production:

BB→X , mB→X, mm→X

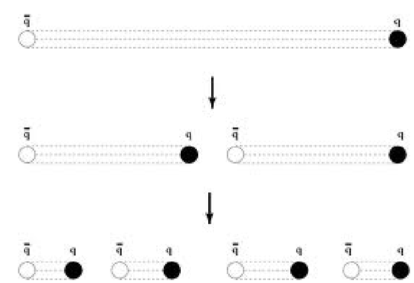
X =many particles

described by

string formation and decay

(string = excited color singlet states *q-qq*, *q-qbar*)

using LUND string model



Covariant transport equation

From non-relativistic to relativistic formulation of transport equations:

Non-relativistic Schrödinger equation → relativistic Dirac equation

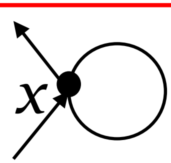
Non-relativistic dispersion relation:	**Relativistic dispersion relation:**
$$E = \frac{\vec{p}^{\,2}}{2m} + U(\vec{r})$$	$$E^{*2} = m^{*2} + \vec{p}^{\,*2}$$ $$m^* = m + U_S$$ $$\vec{p}^{\,*} = \vec{p} + \vec{U}_V$$ $$E^* = E - U_0$$
U(r) **– density dependent potential (with attractive and repulsive parts)**	**U_S – scalar potential (attractive)** **$U_\mu = (U_0, \vec{U}_V)$ vector 4-potential (repulsive)** $\mu = 0,1,2,3$
! Not Lorentz invariant, i.e. dependent on the frame	**! Lorentz invariant, i.e. independent on the frame**

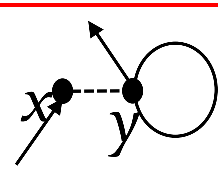
➔ Consider the Dirac equation with local and non-local mean fields:

$$(i\gamma^\mu \partial_\mu - m)\psi(x) - U^{MF}(x)\psi(x) - \int d^4y\, U^{MD}(x,y)\psi(x) = 0$$

here

$x \equiv (t, \vec{r}) \quad y \equiv (t, \vec{r}\,')$





Covariant transport equation

- **Covariant relativistic on-shell BUU equation :**

  from many-body theory by connected Green functions in phase-space + mean-field limit for the propagation part (VUU)

$$\left\{ \left(\Pi_\mu - \Pi_\nu (\partial_\mu^p U_V^\nu) - m^* (\partial_\mu^p U_S^\nu) \right) \partial_x^\mu + \left(\Pi_\nu (\partial_\mu^x U_V^\nu) + m^* (\partial_\mu^x U_S^\nu) \right) \partial_p^\mu \right\} f(x,p) = I_{coll}$$

$$I_{coll} \equiv \sum_{2,3,4} \int d2\; d3\; d4\;\; [G^+ G]_{1+2 \to 3+4}\; \delta^4 (\Pi + \Pi_2 - \Pi_3 - \Pi_4)$$

$$d2 \equiv \frac{d^3 p_2}{E_2}$$

$$\times \{ f(x,p_3)\; f(x,p_4)\; (1 - f(x,p))\; (1 - f(x,p_2))$$

$$- f(x,p)\; f(x,p_2)\; (1 - f(x,p_3))\; (1 - f(x,p_4)) \}$$

Gain term $3+4 \rightarrow 1+2$

Loss term $1+2 \rightarrow 3+4$

where $\partial_\mu^x \equiv (\partial_t, \vec{\nabla}_r)$

m*(x,p) = m + U_S(x,p) - effective mass

Π_μ(x,p) = p_μ – U_μ(x,p) - effective momentum

U_S(x,p), U_μ(x,p) **are scalar and vector part of particle self-energies**

$\delta(\Pi_\mu \Pi^\mu - m^{*2})$ **– mass-shell constraint**

W. Ehehalt, W. Cassing, Nucl. Phys. A 602 (1996) 449

Brueckner theory

Transition rate for the process ***1+2→3+4*** follows **from many-body Brueckner theory:**

$$[G^+G]_{1+2\to 3+4}\ \delta^4(\Pi+\Pi_2-\Pi_3-\Pi_4)$$

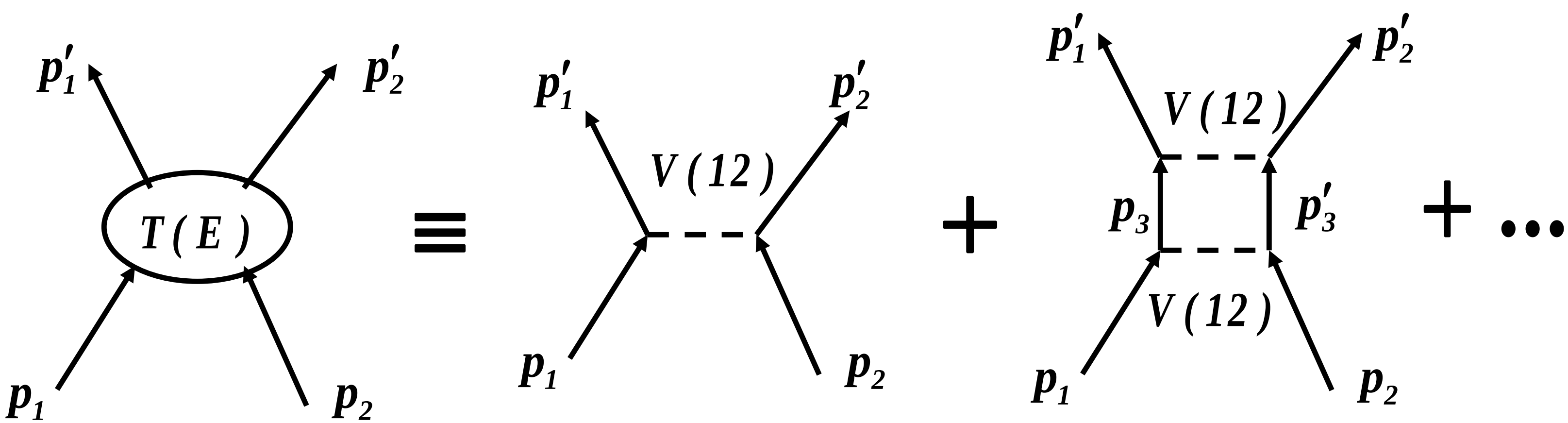
1) 2-body scattering in vacuum:

Scattering amplitude:

$$T(E)=V+V\frac{1}{E-t(1)-t(2)+i\eta}T(E)$$

with the hamiltonian:

$$H=\sum_{i=1}^{A}t(i)+\frac{1}{2}\sum_{i<j}V(ij)$$



‚ladder' resummation

Brueckner theory

2) 2-body scattering in the medium:

Scattering amplitude → from Brueckner theory:

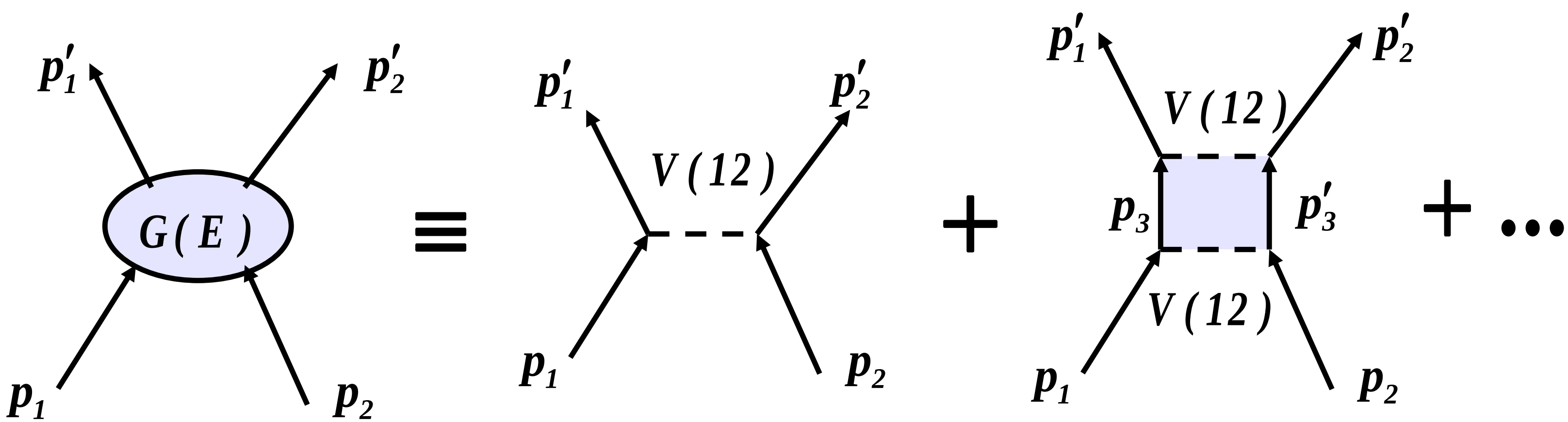
$$G(E) = V + V\frac{1}{E - h(1) - h(2) + i\eta}(1 - n_3 - n_3')\,G(E)$$

Pauli-blocking

n_3 – **occupation number**

with single-particle hamiltonian: $h(1) = t(1) + U^{MF}(1)$

Note: vacuum case : $h(1) = t(1)$ and $n_3 = n_3' = 0 \Rightarrow G-matrix \to T-matrix$

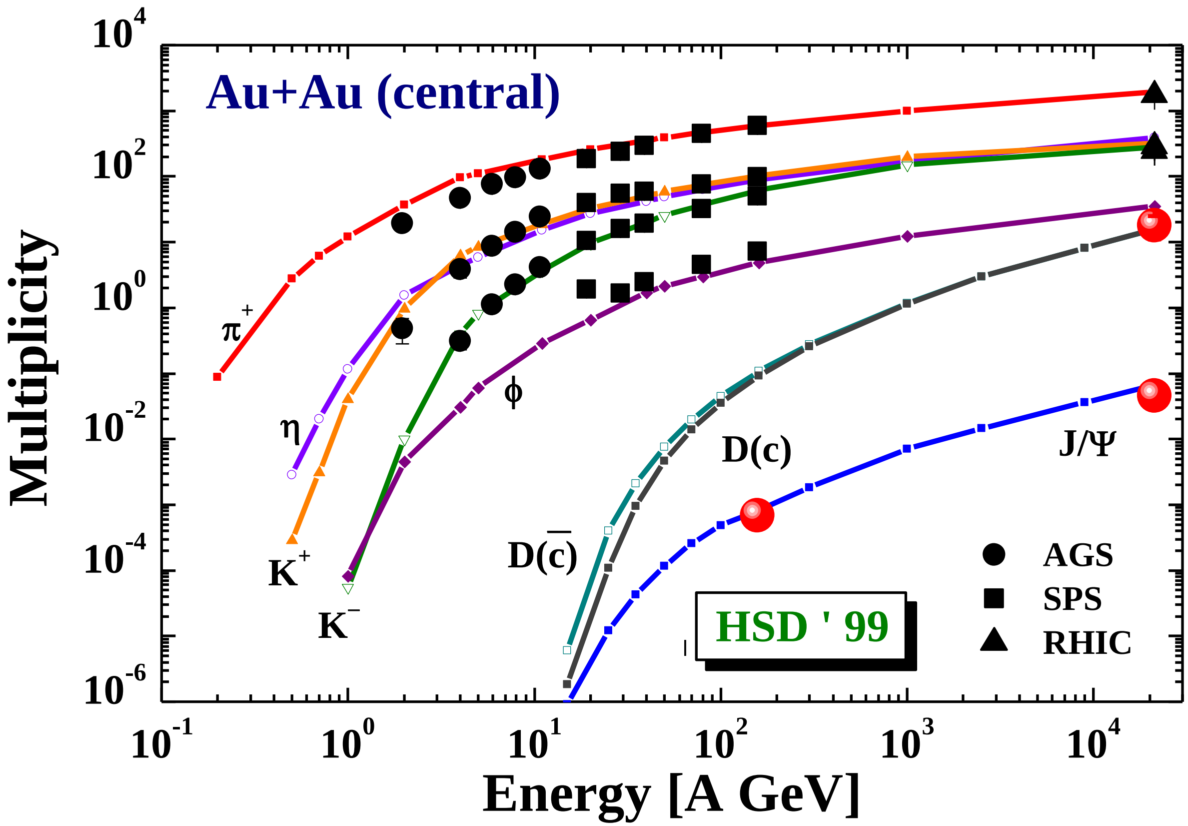


Propagation between scattering *V(12)* with mean field hamiltonian *h(1), h(2)*

! only allowed if intermediate states *3,3'* are not accupied !

Hadron-String-Dynamics – a microscopic transport model for heavy-ion reactions

- very good description of particle production in pp, pA, pA, AA reactions
- unique description of nuclear dynamics from low (~100 MeV) to ultrarelativistic (>20 TeV) energies



HSD predictions from 1999; data from the new millenium

W. Cassing, E.B., Phys.Rept. 308 (1999) 65