2. Density-matrix formalism: Correlation dynamics

Density-matrix formalism

- **Schrödinger equation for a system of N fermions:**

$$i\hbar\frac{\partial}{\partial t}\,\Psi_N(1,..,N;t) = H_N(1,..,N)\Psi_N(1,..,N;t) \qquad (1)$$

or in Dirac notation:

$$i\hbar\frac{\partial}{\partial t}\,|\Psi_N(t)>= H_N|\Psi_N(t)>$$

- **Hamiltonian operator:**

$$H_N = \sum_{i=1}^{N} h^0(i) + \sum_{i<j}^{N-1} v(ij),$$

2-body interaction (potential)

One-body Hamiltonian:

$$h^0(i) = t(i) + U^0(i)$$

kinetic energy operator + (possible) external mean-field potential (e.g. external electromagnetic field)

Hermitian Hamiltonian: $H_N = H_N^\dagger$

Notation:
***i* – particle index of many body system (*i=1,…,N*) :**

$$i \equiv \mathbf{r}_i, \sigma_i, \tau_i$$

***I* = coordinate, spin, isospin,...**

Consider (1) - hermitean conjugate:

$$-i\hbar\frac{\partial}{\partial t'}\,\Psi_N^*(1',..,N';t) = H_N(1',..,N'),\Psi_N^*(1',..,N';t') \qquad (2)$$

(1)*Ψ*$_N$ – Ψ$_N$ (2) :

$$i\hbar\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial t'}\right)\;\Psi_N(1,..,N;t)\;\Psi_N^*(1',..,N';t') =$$

$$(H_N(1,..,N) - H(1',..,N';t'))\,\Psi_N(1,..,N;t)\Psi_N^*(1',..,N';t') \qquad (2.1)$$

Density-matrix formalism

- Introduce two time density :

$$\rho_N(1,..,N,1',..,N';t,t') = \Psi_N(1,..,N;t)\Psi_N^*(1',..,N';t') \qquad (2.2)$$

or in Dirac notation:

$$\rho_N(1,..,N,1',..,N';t,t') = \langle 1',..,N'|\rho_N(t,t')|1,..,N\rangle$$

Substitute (2.2) in (2.1):

$$i\hbar\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial t'}\right)\rho_N = \left(H_N(1,..,N) - H(1',..,N';t')\right)\rho_N$$

$$i\hbar\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial t'}\right)\rho_N = [H_N,\rho_N]$$

- **Restrict to *t'=t* :**

$$\boxed{\rho_N(1,..,N,1',..,N';t) = \rho_N(1,..,N,1',..,N';t,t')\ \delta(t-t')}$$

- **Schrödinger eq. in density-operator representation →**

von Neumann (or Liouville) eq. (in matrix representation) describes an N-particle system in- or out-off equilibrium :

$$\boxed{i\hbar\frac{\partial}{\partial t}\rho_N(1,..,N;1'..N';t) = [H_N,\rho_N(1,..,N;1'..N';t)]} \qquad (2.3)$$

Density-matrix formalism

- Introduce a **reduced density matrices** $\rho_n(1\ldots n, 1'\ldots n'; t)$ **by taking the trace (integrate) over particles** $n+1,\ldots N$ **of** ρ_N**:**

$$\rho_n = \frac{1}{(N-n)!}\mathrm{Tr}_{n+1,\ldots,N}\rho_N = \frac{1}{n+1}\mathrm{Tr}_{n+1}\{\rho_{n+1}\} \quad (3)$$

Recurrence

Here the relative normalization between ρ_n **and** ρ_{n+1} **is fixed and it is useful to choose the normalization**

$$Tr_{1,\ldots,N}\ \rho_N = N!$$

which leads to the following normalization for the one-body density matrix:

$$Tr_{(1=1')}\rho(11';t) = \sum_i < a_i^\dagger a_i >= N$$

$$\mathrm{Tr} \to \int \frac{d^3p}{(2\pi\hbar)^3}\int d^3r$$

i.e. the particle number for the *N*-body Fermi system.

The normalization of the two-body density matrix then reads as

$$Tr_{(1,2)}\rho_2 = \sum_{i,j} < a_i^\dagger a_j^\dagger a_j a_i >= -\sum_{i,j} < a_i^\dagger a_j^\dagger a_i a_j >= \sum_{i,j}\{< a_i^\dagger a_i a_j^\dagger a_j > - < a_i^\dagger a_j > \delta_{ij}\}$$

$$= (N-1)\sum_j < a_j^\dagger a_j >= N(N-1)$$

The traces of the density matrices ρ_n **(for n < N) :** $Tr_{(1,..,n)}\rho_n = \dfrac{N!}{(N-n)!}$

Density matrix formalism: BBGKY-Hierarchy

Taking corresponding traces (i.e. $Tr_{(n+1,\dots N)}$) of the von-Neumann equation we obtain the BBGKY-Hierarchy (Bogolyubov, Born, Green, Kirkwood and Yvon)

$$i\frac{\partial}{\partial t}\,\rho_n = [\sum_{i=1}^{n} h^0(i), \rho_n] + [\sum_{1=i\langle j}^{n-1} v(ij), \rho_n] + \sum_{i=1}^{n} \mathrm{Tr}_{n+1}[v(i, n+1), \rho_{n+1}] \qquad (4)$$

for $1 \le n \le N$ *with* $\rho_{N+1} = 0.$

- **This set of equations is equivalent to the von-Neumann equation**
- **The approximations or truncations of this set will reduce the information about the system**

❑ **The explicit equations for *n=1, n=2* read:**

$$i\frac{\partial}{\partial t}\,\rho_1 = [h^0(1), \rho_1] + \mathrm{Tr}_2[v(12), \rho_2], \qquad (5)$$

$$i\frac{\partial}{\partial t}\,\rho_2 = [\sum_{i=1}^{2} h^0(i), \rho_2] + [v(12), \rho_2] + \mathrm{Tr}_3[v(13) + v(23), \rho_3] \qquad (6)$$

Eqs. (5,6) are not closed since eq. (6) for ρ_2 requires information from ρ_3. Its equation reads:

$$i\frac{\partial}{\partial t}\,\rho_3 = [\sum_{i=1}^{3} h^0(i), \rho_3] + [v(12) + v(13) + v(23), \rho_3] + Tr_4[v(14) + v(24) + v(34), \rho_4] \qquad (7)$$

Correlation dynamics

Introduce the cluster expansion ➔ Correlation dynamics:

- 1-body density matrix: $\rho_1(11') = \rho(11'),$

1 – initial state of particle „1“
1‘ – final state of the same particle „1“

- 2-body density matrix (consider fermions):

(8) $$\rho_2(12,1'2') = \rho(11')\rho(22') - \rho(12')\rho(21') + c_2(12,1'2') = \rho_{20}(12,1'2') + c_2(12,1'2')$$

2PI= 2-particle-irreducible approach

(9) $$\rho_2(12,1'2') = \mathcal{A}_{12}\rho(11')\rho(22') + c_2(12,1'2')$$

2-body antisymmetrization operator:

$$\mathcal{A}_{ij} = 1 - P_{ij}$$

Permutation operator

1PI = 1-particle-irreducible approach (TDHF approximation) + 2-body correlations

By neglecting c_2 in (9) we get the limit of independent particles (Time-Dependent Hartree-Fock). This implies that all effects from collisions or correlations are incorporated in c_2 and higher orders in c_2 etc.

- 3-body density matrix:

$$\rho_3(123,1'2'3') = \rho(11')\rho(22')\rho(33') - \rho(12')\rho(21')\rho(33')$$
$$-\rho(13')\rho(22')\rho(31') - \rho(11')\rho(32')\rho(23') + \rho(13')\rho(21')\rho(32') + \rho(12')\rho(31')\rho(23')$$
$$+\rho(11')c_2(23,2'3') - \rho(12')c_2(23,1'3') - \rho(13')c_2(23,2'1') + \rho(22')c_2(13,1'3')$$
$$-\rho(21')c_2(13,2'3') - \rho(23')c_2(13,1'2') + \rho(33')c_2(12,1'2') - \rho(31')c_2(12,3'2')$$
$$-\rho(32')c_2(12,1'3') + c_3(123,1'2'3'). \quad (10)$$

3-body correlations

Correlation dynamics

The goal: from BBGKY-Hierarchy obtain the closed equation **for 1-body density matrix** within 2PI **discarding explicit three-body correlations c_3**

- **for that we reformulate eq. (5) for ρ_1 using cluster expansion (correlation dynamics):**

$$i\hbar\frac{\partial}{\partial t}\rho_1 = [h^0(1), \rho_1] + Tr_2[v(12), \rho_2] \qquad (5)$$

substitutute eq. (8) for ρ_2

$$(8) \qquad \rho_2(12, 1'2') = \rho(11')\rho(22') - \rho(12')\rho(21') + c_2(12, 1'2') = \rho_{20}(12, 1'2') + c_2(12, 1'2')$$

➔ **we obtain EoM for the one-body density matrix:**

$$i\frac{\partial}{\partial t}\rho(11';t) = [h^0(1) - h^0(1')]\rho(11';t)$$
$$+\mathrm{Tr}_{(2=2')}[v(12)\mathcal{A}_{12} - v(1'2')\mathcal{A}_{1'2'}]\rho(11';t)\rho(22';t) + \mathrm{Tr}_{(2=2')}[v(12) - v(1'2')]c_2(12, 1'2';t)$$

2-body correlations

*** How to obtain the 2-body correlation matrix c_2 ?**

Correlation dynamics

- In order to obtain the 2-body correlation matrix c_2, we start with eq. (6) for ρ_2

(6) $$i\hbar\frac{\partial}{\partial t}\rho_2 = [\sum_{i=1}^{2} h^0(i), \rho_2] + [v(12), \rho_2] + Tr_3[v(13) + v(23), \rho_3]$$

$$\rho_3(123, 1'2'3') = \rho(11')\rho(22')\rho(33') - \rho(12')\rho(21')\rho(33')$$ (10)

$$+\rho(13')\rho(22')\rho(31') - \rho(11')\rho(32')\rho(23') + \rho(13')\rho(21')\rho(32') + \rho(12')\rho(31')\rho(23')$$
$$+\rho(11')c_2(23, 2'3') - \rho(12')c_2(23, 1'3') - \rho(13')c_2(23, 2'1') + \rho(22')c_2(13, 1'3')$$
$$-\rho(21')c_2(13, 2'3') - \rho(23')c_2(13, 1'2') + \rho(33')c_2(12, 1'2') - \rho(31')c_2(12, 3'2')$$
$$-\rho(32')c_2(12, 1'3') + c_3(123, 1'2'3')$$

substitute eq. (10) for ρ_3
and discarding explicit 3-body correlations c_3

➔ we obtain EoM for the two-body correlation matrix c_2 :

$$i\frac{\partial}{\partial t}c_2(12, 1'2'; t) = [h^0(1) + h^0(2) - h^0(1') - h^0(2')]c_2(12, 1'2'; t)$$ (12)

$$+\mathrm{Tr}_{(3=3')}[v(13)\mathcal{A}_{13} + v(23)\mathcal{A}_{23} - v(1'3')\mathcal{A}_{1'3'} - v(2'3')\mathcal{A}_{2'3'}]\rho(33'; t)c_2(12, 1'2'; t)$$
$$+[v(12) - v(1'2')]\rho_{20}(12, 1'2')$$
$$-\mathrm{Tr}_{(3=3')}\{v(13)\rho(23'; t)\rho_{20}(13, 1'2'; t) - v(1'3')\rho(32'; t)\rho_{20}(12, 1'3'; t)$$
$$+v(23)\rho(13'; t)\rho_{20}(32, 1'2'; t) - v(2'3')\rho(31'; t)\rho_{20}(12, 3'2'; t)\}$$

$$\rho(11')\rho(22') - \rho(12')\rho(21') = \rho_{20}(12, 1'2')$$

$$+[v(12) - v(1'2')]c_2(12, 1'2'; t)$$
$$-\mathrm{Tr}_{(3=3')}\{v(13)\rho(23'; t)c_2(13, 1'2'; t) - v(1'3')\rho(32'; t)c_2(12, 1'3'; t)$$
$$+v(23)\rho(13'; t)c_2(32, 1'2'; t) - v(2'3')\rho(31'; t)c_2(12, 3'2'; t)\}$$

$$+\mathrm{Tr}_{(3=3')}\{[v(13)\mathcal{A}_{13}\mathcal{A}_{1'2'} - v(1'3')\mathcal{A}_{1'3'}\mathcal{A}_{12}]\ \rho(11'; t)c_2(32, 3'2'; t)$$
$$+[v(23)\mathcal{A}_{23}\mathcal{A}_{1'2'} - v(2'3')\mathcal{A}_{2'3'}\mathcal{A}_{12}]\ \rho(22'; t)c_2(13, 1'3'; t)\}.$$

Correlation dynamics

To reduce the complexity we introduce notations:

- a **one-body Hamiltonian** by

$$h(i) = t(i) + U^s(i) = t(i) + \mathrm{Tr}_{(n=n')} v(in)\mathcal{A}_{in}\rho(nn';t), \quad (13)$$

$$h(i') = t(i') + U^s(i') = t(i') + \mathrm{Tr}_{(n=n')} v(i'n')\mathcal{A}_{i'n'}\rho(nn';t)$$

kinetic term + interaction with the self-generated time-dependent mean field

- **Pauli-blocking operator is uniquely defined by** (14)

$$Q_{ij}^{=} = 1 - \mathrm{Tr}_{(n=n')}(P_{in} + P_{jn})\rho(nn';t); \qquad Q_{i'j'}^{=} = 1 - \mathrm{Tr}_{(n=n')}(P_{i'n'} + P_{j'n'})\rho(nn';t),$$

- **Effective 2-body interaction in the medium:**

$$V^{=}(ij) = Q_{ij}^{=} v(ij); \qquad V^{=}(i'j') = Q_{i'j'}^{=} v(i'j'), \quad (15)$$

Resummed interaction ➔ G-matrix approach

Correlation dynamics

❑* EoM for the **one-body density matrix**:

$$i\frac{\partial}{\partial t}\rho(11';t) = [h(1) - h(1')]\rho(11';t) + \mathrm{Tr}_{(2=2')}[v(12) - v(1'2')]c_2(12,1'2';t) \quad (16)$$

TDHF

2-body correlations

EoM (16) describes the propagation of a particle in the self-generated mean field $U^s(i)$ with additional 2-body correlations that are further specified in the EoM (17) for c_2 :

❑* EoM for the **2-body correlation matrix**:

$$i\frac{\partial}{\partial t}c_2(12,1'2';t) = [\sum_{i=1}^{2} h(i) - \sum_{i'=1'}^{2'} h(i')]c_2(12,1'2';t)$$

Propagation of two particles 1 and 2 in the mean field U^s

$$+[V^{=}(12) - V^{=}(1'2')]\rho_{20}(12,1'2';t)$$

Born term: bare 2-body scattering

$$(17) \quad +[V^{=}(12) - V^{=}(1'2')]c_2(12,1'2';t)$$

resummed in-medium interaction with intermediate Pauli blocking (G-matrix theory)

$$+\mathrm{Tr}_{(3=3')}\{[v(13)\mathcal{A}_{13}\mathcal{A}_{1'2'} - v(1'3')\mathcal{A}_{1'3'}\mathcal{A}_{12}]\ \rho(11';t)c_2(32,3'2';t)$$
$$+[v(23)\mathcal{A}_{23}\mathcal{A}_{1'2'} - v(2'3')\mathcal{A}_{2'3'}\mathcal{A}_{12}]\ \rho(22';t)c_2(13,1'3';t)\}.$$

2-Particle-2-hole interactions (important for groundstate correlations) and damping of low energy modes

Note: Time evolution of c_2 depends on the distribution of a third particle, which is integrated out in the trace! The third particle is interacting as well!

*: EoM is obtained after the ‚cluster' expansion and neglecting the explicit 3-body correlations c_3

Vlasov equation

BBGKY-Hierarchie - 1PI ➔
eq.(11) with $c_2(1,2,1',2')=0$

$$i\hbar\frac{\partial}{\partial t}\rho(11';t)=[h^0(1)-h^0(1')]\rho(11';t)$$

TDHF

$$\frac{\partial}{\partial t}\rho(\vec{r},\vec{r}',t)+\frac{i}{\hbar}\left[\frac{\hbar^2}{2m}\vec{\nabla}_r^2+U(\vec{r},t)-\frac{\hbar^2}{2m}\vec{\nabla}_{r'}^2-U(\vec{r}',t)\right]\rho(\vec{r},\vec{r}',t)=0$$

- **perform Wigner transformation of one-body density distribution function ρ(r,r',t)➔**

$$f(\vec{r},\vec{p},t)=\int d^3s\ \exp\left(-\frac{i}{\hbar}\vec{p}\vec{s}\right)\rho\left(\vec{r}+\frac{\vec{s}}{2},\vec{r}-\frac{\vec{s}}{2},t\right) \quad (18)$$

f(r,p,t) **is the single particle phase-space distribution function**

After the 1st order gradient expansion ➔ Vlasov equation of motion
- free propagation of particles in the self-generated HF mean-field potential *U(r,t):*

$$\frac{\partial}{\partial t}f(\vec{r},\vec{p},t)+\frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}}\ f(\vec{r},\vec{p},t)-\vec{\nabla}_{\vec{r}}U(\vec{r},t)\ \vec{\nabla}_{\vec{p}}f(\vec{r},\vec{p},t)=0 \quad (19)$$

$$U(\vec{r},t)=\frac{1}{(2\pi\hbar)^3}\sum_{\beta_{occ}}\int d^3r'd^3p\,V(\vec{r}-\vec{r}',t)\,f(\vec{r}',\vec{p},t)$$

Uehling-Uhlenbeck equation: collision term

$$i\frac{\partial}{\partial t}\rho(11';t) = [h(1) - h(1')]\rho(11';t) + \mathrm{Tr}_{(2=2')}[v(12) - v(1'2')]c_2(12,1'2';t) \quad (21)$$

TDHF – Vlasov equation

2-body correlations

Collision term:

$$I(11',t) = \mathrm{Tr}_{(2=2')}[v(12) - v(1'2')]c_2(12,1'2';t) \quad (22)$$

- perform Wigner transformation
- Formally solve the EoM for c_2 (with some approximations in momentum space):

$$\begin{aligned} i\frac{\partial}{\partial t}c_2(12,1'2';t) &= [\sum_{i=1}^{2} h(i) - \sum_{i'=1'}^{2'} h(i')]c_2(12,1'2';t) \\ &+[V^{=}(12) - V^{=}(1'2')]\rho_{20}(12,1'2';t) \\ &+[V^{=}(12) - V^{=}(1'2')]c_2(12,1'2';t) \\ &+\mathrm{Tr}_{(3=3')}\{[v(13)\mathcal{A}_{13}\mathcal{A}_{1'2'} - v(1'3')\mathcal{A}_{1'3'}\mathcal{A}_{12}]\ \rho(11';t)c_2(32,3'2';t) \\ &+[v(23)\mathcal{A}_{23}\mathcal{A}_{1'2'} - v(2'3')\mathcal{A}_{2'3'}\mathcal{A}_{12}]\ \rho(22';t)c_2(13,1'3';t)\}. \end{aligned} \quad (23)$$

- and insert obtained c_2 in the expression (22) for $I(11',t)$: → BUU EoM

Boltzmann (Vlasov)-Uehling-Uhlenbeck (B(V)UU) equation : Collision term

$$\frac{d}{dt} f(\vec{r},\vec{p},t) \equiv \frac{\partial}{\partial t} f(\vec{r},\vec{p},t) + \frac{\vec{p}}{m}\vec{\nabla}_{\vec{r}}\, f(\vec{r},\vec{p},t) - \vec{\nabla}_{\vec{r}} U(\vec{r},t)\, \vec{\nabla}_{\vec{p}} f(\vec{r},\vec{p},t) = \left(\frac{\partial f}{\partial t}\right)_{coll} \qquad (24)$$

Collision term for 1+2→3+4 (let's consider fermions) :

$$I_{coll} = \frac{4}{(2\pi)^3}\int d^3 p_2\, d^3 p_3 \int d\Omega\ |\upsilon_{12}|\, \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\cdot \frac{d\sigma}{d\Omega}(1+2 \rightarrow 3+4)\cdot P \qquad (25)$$

Probability including Pauli blocking of fermions:

$$P = f_3 f_4 (1-f_1)(1-f_2) - f_1 f_2 (1-f_3)(1-f_4) \qquad (26)$$

Gain term (first term): **3+4→1+2**

Loss term (second term): **1+2→3+4**

For particle 1 and 2:
Collision term = Gain term – Loss term

$$I_{coll} = G - L$$

The BUU equations (24) describes the propagation in the self-generated mean-field U(r,t) as well as mutual two-body interactions respecting the Pauli-principle

2. Quantum field theory

➔ **Kadanoff-Baym dynamics**

➔ **generalized off-shell transport equations**

From weakly to strongly interacting systems

In-medium effects (on hadronic or partonic levels!) = changes of particle properties in the hot and dense medium

Example: hadronic medium - vector mesons, strange mesons

QGP – dressing of partons

Many-body theory:

Strong interaction ➔ **large width** = short life-time

➔ broad spectral function ➔ **quantum object**

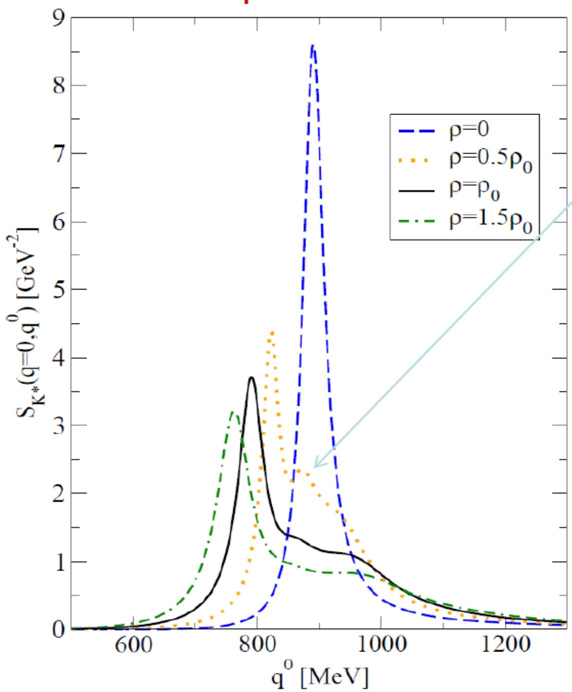
- **How to describe the dynamics of broad strongly interacting quantum states in transport theory?**

❑ **semi-classical BUU**

first order gradient expansion of quantum **Kadanoff-Baym equations**

❑ **generalized transport equations based on Kadanoff-Baym dynamics**

Kbar* spectral function



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Dynamical description of strongly interacting systems

- **Semi-classical on-shell BUU:** applies for small collisional width, i.e. for a weakly interacting systems of particles

How to describe strongly interacting systems?!

- **Quantum field theory** ➔
Kadanoff-Baym dynamics for **resummed** single-particle Green functions $S^<$ (= $G^<$)

(1962)

$$\hat{S}_{0x}^{-1} \, S_{xy}^{<} = \Sigma_{xz}^{ret} \odot S_{zy}^{<} + \Sigma_{xz}^{<} \odot S_{zy}^{adv}$$

Integration over the intermediate spacetime

Green functions $S^<$ / self-energies Σ:

$$iS_{xy}^{<} = \eta \langle \{ \Phi^{+}(y) \Phi(x) \} \rangle$$

$$iS_{xy}^{>} = \langle \{ \Phi(y) \Phi^{+}(x) \} \rangle$$

$$iS_{xy}^{c} = \langle T^{c} \{ \Phi(x) \Phi^{+}(y) \} \rangle \quad - causal$$

$$iS_{xy}^{a} = \langle T^{a} \{ \Phi(x) \Phi^{+}(y) \} \rangle \quad - anticausal$$

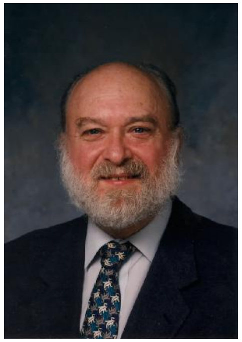
$$S_{xy}^{ret} = S_{xy}^{c} - S_{xy}^{<} = S_{xy}^{>} - S_{xy}^{a} \quad - retarded$$

$$S_{xy}^{adv} = S_{xy}^{c} - S_{xy}^{>} = S_{xy}^{<} - S_{xy}^{a} \quad - advanced$$

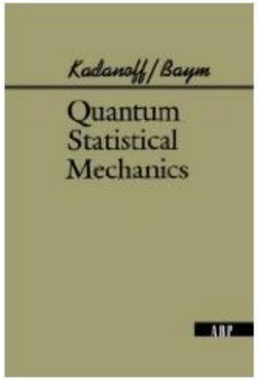
$$\eta = \pm 1 (bosons / \; fermions)$$

$$T^{a}(T^{c}) - (anti-)time - ordering \; operator$$

$$\hat{S}_{0x}^{-1} \equiv -(\partial_x^{\mu} \partial_{\mu}^{x} + M_0^2)$$



Leo Kadanoff



Gordon Baym

Heisenberg picture

❑ Relativistic formulations of the many-body problem are described within covariant field theory.

The fields themselves are distributions in space-time $x = (t, \mathbf{x})$ ➔
from Schrödinger picture → Heisenberg picture:

❑ In the Heisenberg picture the time evolutions of the system is described by time-dependent operators that are evolved with the help of the unitary time-evolution operator *U* (*t, t'*) which follows

$$i\frac{\partial \hat{U}(t,t_0)}{\partial t} = \hat{H}(t)\hat{U}(t,t_0) \tag{1}$$

Schrödinger operator of the system

Eq. (1) has the formal solution:

$$\hat{U}(t,t_0) = T\left(\exp\left[-i\int_{t_0}^{t} \mathrm{dz}\ \hat{\mathrm{H}}(\mathrm{z})\right]\right) = \sum_{n=0}^{\infty} \frac{T[-i\int_{t_0}^{t} \mathrm{dz}\ \hat{\mathrm{H}}(\mathrm{z})]^{\mathrm{n}}}{n!} \tag{2}$$

If *H* doesn't depend on time: $\hat{U}(t,t_0) = e^{-i\hat{H}(t-t_0)}$

$$\Psi(x,t) = \hat{U}(t,t_0=0)\Psi(x,t_0=0)$$

Elementary observables in Heisenberg picture

- The time evolution of any operator *O* in the Heisenberg picture from time t_0 to *t* is given by

$$\hat{O}_H(t) = \hat{U}^+(t,t_o)\,\hat{O}\,\hat{U}(t,t_o) \tag{3}$$

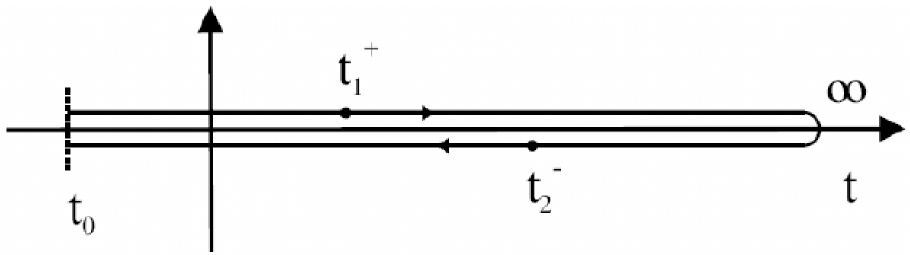
$$\Rightarrow \quad \hat{O}_H(t) = e^{iH(t-t_0)}\hat{O}\; e^{-iH(t-t_0)}$$

- If the initial state is given by some density matrix *ρ*, which may be a pure or mixed state, then the time evolution of expectation value *O(t)* of the operator *O* in the Heisenberg picture from time t_0 to *t* is given by

(4)

$$O(t) = \langle \hat{O}_H(t)\rangle = \mathrm{Tr}\left(\hat{\rho}\,\hat{\mathrm{O}}_{\mathrm{H}}(\mathrm{t})\right) = \mathrm{Tr}\left(\hat{\rho}\,\hat{\mathrm{U}}(\mathrm{t}_0,\mathrm{t})\hat{\mathrm{O}}\,\hat{\mathrm{U}}(\mathrm{t},\mathrm{t}_0)\right) = \mathrm{Tr}\left(\hat{\rho}\,\hat{\mathrm{U}}^{\dagger}(\mathrm{t},\mathrm{t}_0)\hat{\mathrm{O}}\,\hat{\mathrm{U}}(\mathrm{t},\mathrm{t}_0)\right)$$

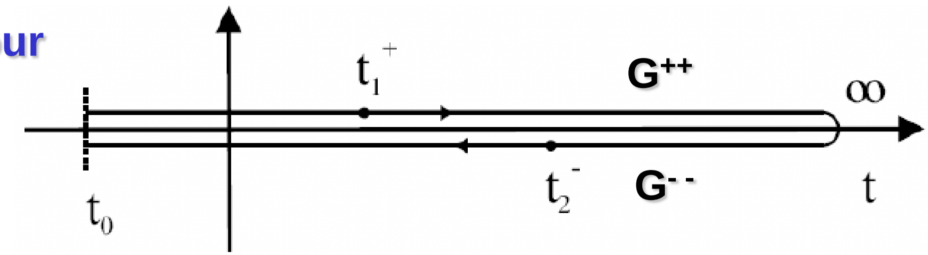
This implies that first the system is evolved from t_0 to t and then backward from t to t_0. This may be expressed as a time integral along the Keldysh-Contour



Two-point functions on the Keldysh contour

Real-time (Keldysh-) Contour

in the Heisenberg picture



Consider: Interacting field theory for spinless massive scalar bosons ➔ scalar field $\phi(x)$

- **Green functions: elementary degrees of freedom** $x = (t_x, \vec{x}), \quad y = (t_y, \vec{y})$

Causal: $iG^c(x,y) = iG^{++}(x,y) = \langle\, \hat{T}^c(\phi(x)\phi(y))\,\rangle$ **t_y and t_x on upper part; $t_x > t_y$**

Small: $iG^<(x,y) = iG^{+-}(x,y) = \langle \phi(y)\phi(x)\rangle$ **t_y on upper; t_x on lower part**

Large: $iG^>(x,y) = iG^{-+}(x,y) = \langle \phi(x)\phi(y)\rangle$ **t_y on lower; t_x on upper part**

Anticausal: $iG^a(x,y) = iG^{--}(x,y) = \langle\, \hat{T}^a(\phi(x)\phi(y))\,\rangle$ **t_y and t_x on lower part; $t_y > t_x$**

T^c / T^a denote time ordering on the upper/lower branch of the real-time contour

In matrix notation:

$$G(x,y) = \begin{array}{c} \\ + \\ - \end{array} \begin{array}{c} \quad + \qquad\qquad - \\ \begin{pmatrix} G^c(x,y) & G^<(x,y) \\ G^>(x,y) & G^a(x,y) \end{pmatrix} \end{array} \qquad (5)$$

Green functions on contour

- Relation to the one-body density matrix ρ :

(6) $$\rho(\mathrm{x},\mathrm{x}';t) = -iG^<(\mathrm{x},\mathrm{x}';t,t)$$

$$G^<(\mathrm{x},\mathrm{x}';t) = \int_{-\infty}^{\infty} d(\tau-\tau')\ G^<(\mathrm{x},\mathrm{x}';\tau,\tau')$$

$t=(\tau+\tau')/2$

- **Two-point functions *F* on the closed-time-path (CTP) generally can be expressed by retarded (R) and advanced (A) components as**

(7)
$$F^{\mathrm{R}}(x,y) = F^c(x,y) - F^<(x,y) = F^>(x,y) - F^a(x,y)$$
$$F^{\mathrm{A}}(x,y) = F^c(x,y) - F^>(x,y) = F^<(x,y) - F^a(x,y)$$

Note:
only two Green functions are independent!

giving in particular the relation

(8) $$F^{\mathrm{R}}(x,y) - F^{\mathrm{A}}(x,y) = F^>(x,y) - F^<(x,y)$$

Note that the advanced and retarded components of the Green functions contain only spectral and no statistical information (see below)

Dyson-Schwinger equation on the contour

❑ Dyson-Schwinger equation:

$$G(x,y) = G_0(x,y) + G_0(x,y)\Sigma(x,y)G(x,y) \qquad (9)$$

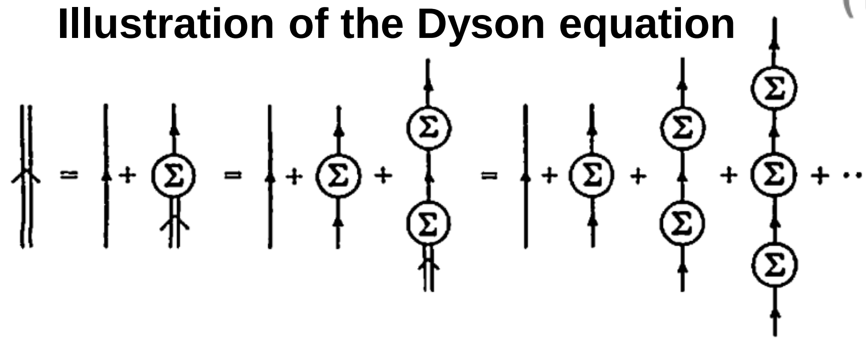
$$\hat{G}_{0x}^{-1} = -(\partial_\mu^x \partial_x^\mu + m^2)$$

Dyson-Schwinger equation on the closed-time-path reads in matrix form:

$$\begin{pmatrix} G^c(x,y) & G^<(x,y) \\ G^>(x,y) & G^a(x,y) \end{pmatrix} = \begin{pmatrix} G_0^c(x,y) & G_0^<(x,y) \\ G_0^>(x,y) & G_0^a(x,y) \end{pmatrix} +$$
$$\begin{pmatrix} G_0^c(x,x') & G_0^<(x,x') \\ G_0^>(x,x') & G_0^a(x,x') \end{pmatrix} \odot \begin{pmatrix} \Sigma^c(x',y') & -\Sigma^<(x',y') \\ -\Sigma^>(x',y') & \Sigma^a(x',y') \end{pmatrix} \odot \begin{pmatrix} G^c(y',y) & G^<(y',y) \\ G^>(y',y) & G^a(y',y) \end{pmatrix} \qquad (10)$$

The selfenergy Σ on the CTP is defined along (9) and incorporates interactions of higher order. In lowest order Σ/2*M* is given by the Hartree or Hartree-Fock mean field but it follows a nonperturbative expansion

Illustration of the Dyson equation



$\odot_p$ **means convolution integral over the closed time-path**

Towards the Kadanoff-Baym equations

For Bose case the free propagator is defined via the negative inverse Klein-Gordon operator in space-time representation

$$\hat{G}_{0x}^{-1} = -(\partial_\mu^x \partial_x^\mu + m^2) \qquad (11)$$

which is a solution of the Klein-Gordon equation in the following sense:

$$\boxed{\hat{G}_{0x}^{-1} G_0^{R/A}(x,y) = \delta(x-y)}$$

$$\hat{G}_{0x}^{-1} \begin{pmatrix} G_0^c(x,y) & G_0^<(x,y) \\ G_0^>(x,y) & G_0^a(x,y) \end{pmatrix} = \delta(\mathbf{x}-\mathbf{y}) \begin{pmatrix} \delta(x_0-y_0) & 0 \\ 0 & -\delta(x_0-y_0) \end{pmatrix} \qquad (12)$$

Free Green function $G_0(x,y)$

$$= \delta(\mathbf{x}-\mathbf{y})\delta_p(x_0-y_0)$$

$$x = (x^0, \mathbf{x})$$
$$y = (y^0, \mathbf{y})$$

with δ_p denoting the δ-function on the closed time path (CTP).
In (11) *m* denotes the bare mass of the scalar field.

The Kadanoff-Baym equations

To derive the Kadanoff-Baym equations one multiplies Dyson-Schwinger eq. (10) with G_{0x}^{-1}. This gives four equations for $G^<$, $G^>$ which can be written in the form:

1) (10)*G_{0x}^{-1} ➔ propagation of Green functions in variable x

$$-(\partial_\mu^x \partial_x^\mu + m^2) G^{R/A}(x,y) = \delta(x-y) + \Sigma^{R/A}(x,x') \odot G^{R/A}(x',y)$$

$$-(\partial_\mu^x \partial_x^\mu + m^2) G^<(x,y) = \Sigma^R(x,x') \odot G^<(x',y) + \Sigma^<(x,x') \odot G^A(x',y) \qquad (13)$$

$$-(\partial_\mu^x \partial_x^\mu + m^2) G^>(x,y) = \Sigma^R(x,x') \odot G^>(x',y) + \Sigma^>(x,x') \odot G^A(x',y)$$

2) (10)*G_{0y}^{-1} ➔ propagation of Green functions in variable y

$$-(\partial_\mu^y \partial_y^\mu + m^2) G^{R/A}(x,y) = \delta(x-y) + G^{R/A}(x,x') \odot \Sigma^{R/A}(x',y)$$

$$-(\partial_\mu^y \partial_y^\mu + m^2) G^<(x,y) = G^R(x,x') \odot \Sigma^<(x',y) + G^<(x,x') \odot \Sigma^A(x',y) \qquad (14)$$

$$-(\partial_\mu^y \partial_y^\mu + m^2) G^>(x,y) = G^R(x,x') \odot \Sigma^>(x',y) + G^>(x,x') \odot \Sigma^A(x',y)$$

Note: propagation in both variables needed !

- retarded/advanced Green functions only depend on retarded/advanced quantities and contain only spectral information (no information on particle density)!

Derivation of the selfenergy

Effective action Γ :

$$\Gamma[G] = \Gamma^0 + \frac{i}{2}\left[\ln(1 - \odot_p G_0 \odot_p \Sigma) + \odot_p G \odot_p \Sigma\right] + \Phi[G] \quad (15)$$

Resummed propagators with self-generated mean-field

Γ^0 – ‚free' part of action (kinetic + mass terms), G_0 - free propagator,
$\odot_p$ means convolution integral over the closed time-path

$\Phi(G)$ is the ‚interaction part' = sum of all connected nPI diagrams built up by the full G(x,y)

Approximation: Two-particle irreducible (2PI) diagrams

- Define selfenergy Σ by the variation of Γ [G]

$$\delta\Gamma = 0 = \frac{i}{2}\Sigma\,\delta G - \frac{i}{2}\frac{G_0}{1 - G_0\,\Sigma}\delta\Sigma + \frac{i}{2}G\,\delta\Sigma + \delta\Phi \quad (16)$$

$$= \frac{i}{2}\Sigma\,\delta G - \frac{i}{2}\underbrace{\frac{1}{G_0^{-1} - \Sigma}}_{=G}\delta\Sigma + \frac{i}{2}G\,\delta\Sigma + \delta\Phi = \frac{i}{2}\Sigma\,\delta G + \delta\Phi$$

$\Rightarrow$

$$\Sigma = 2i\frac{\delta\Phi}{\delta G} = 2\frac{\delta\Phi}{\delta(-iG)}$$

➔ The selfenergy Σ are obtained by opening of a propagator line in the irreducible diagrams Φ

Example: scalar theory with self-interactions

Φ^4 – theory: the interacting field theory for spinless massive scalar bosons provides a ‚theoretical laboratory' for testing approximation schemes

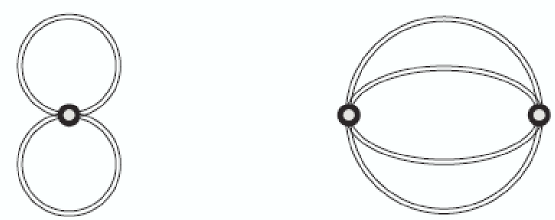
❑ Lagrangian density:

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu^x \phi(x) \partial_x^\mu \phi(x) - \frac{1}{2} m^2 \phi(x)^2 - \frac{\lambda}{4!}\phi^4(x) \qquad (17)$$

$\phi(x)$ – real scalar field
λ – is a coupling constant

❑ $\Phi(G)$: the sum of all closed 2PI diagrams built up by the full G(x,y):

$\Phi(G)$ up to 3-loop order;
~ 2nd order in λ (i.e. 2PI)

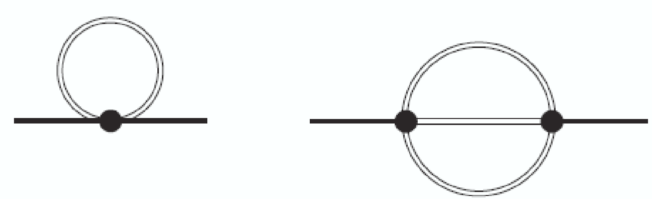


d+1: *d*=dimension of space
(*d*=3 or 2) + 1(=time)

$$i\Phi = \frac{i\lambda}{8}\int_p d^{d+1}x \; G(x,x)^2 - \frac{\lambda^2}{48}\int_p d^{d+1}x \int_p d^{d+1}y \; G(x,y)^4 \qquad (18)$$

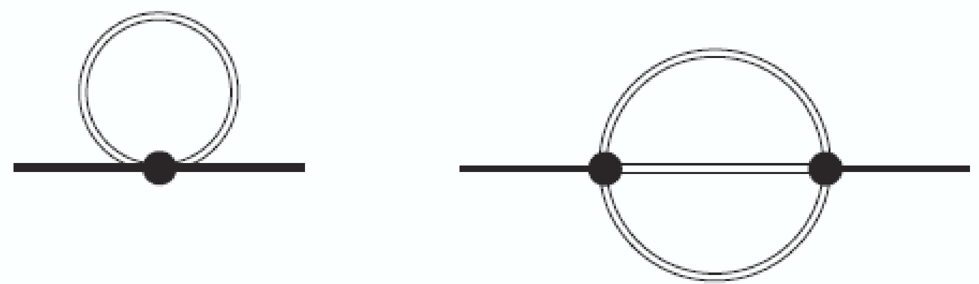
From (16) ➔ self-energies are defined by the variation of Φ w.r.t G(y,x):

$$\Sigma(x,y) = 2i \frac{\delta\Phi}{\delta G(y,x)}$$



➔ Cut a line and stretch:

2PI self-energies in Φ^4 - theory



$$\Sigma(x,y) = \underline{\Sigma^\delta(x)\,\delta_p^{(d+1)}(x-y)} + \underline{\Theta_p(x_0-y_0)\,\Sigma^>(x,y) + \Theta_p(y_0-x_0)\,\Sigma^<(x,y)}$$

Local in space and time part: tadpole

Nonlocal part: sunset

$$\Sigma^\delta(x) = \frac{\lambda}{2}\, i\, G^<(x,x)$$

$$\Sigma^{\gtrless}(x,y) = -\frac{\lambda^2}{6}\, G^{\gtrless}(x,y)\, G^{\gtrless}(x,y)\, G^{\lessgtr}(y,x) = -\frac{\lambda^2}{6}\left[G^{\gtrless}(x,y)\right]^3$$

local ‚potential' term (~λ)
leads to the generation of an effective mass for the field quanta

interaction term (~ λ^2)

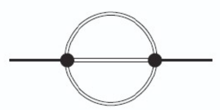
Kadanoff-Baym equations of motion for $G^<$

1) $$-\left[\partial_\mu^x \partial_x^\mu + m^2\right] G^{\gtrless}(x,y) = \underline{\Sigma^\delta(x)\, G^{\gtrless}(x,y)}$$ **potential term**

interaction term

$$+ \int_{t_0}^{x_0} dz_0 \int d^d z \; \left[\Sigma^>(x,z) - \Sigma^<(x,z)\right] G^{\gtrless}(z,y)$$

$$- \int_{t_0}^{y_0} dz_0 \int d^d z \; \Sigma^{\gtrless}(x,z) \left[G^>(z,y) - G^<(z,y)\right],$$



2) $$-\left[\partial_\mu^y \partial_y^\mu + m^2\right] G^{\gtrless}(x,y) = \underline{\Sigma^\delta(y)\, G^{\gtrless}(x,y)}$$

d: dimension of space

$$+ \int_{t_0}^{x_0} dz_0 \int d^d z \; \left[G^>(x,z) - G^<(x,z)\right] \Sigma^{\gtrless}(z,y)$$

$$- \int_{t_0}^{y_0} dz_0 \int d^d z \; G^{\gtrless}(x,z) \left[\Sigma^>(z,y) - \Sigma^<(z,y)\right],$$

Thus, the Kadanoff-Baym equations include the influence of the mean-field on the particle propagation (generated by the tadpole diagram) as well as scattering processes as inherent in the sunset diagram.

☺ **Go ahead and solve KBE with some initial condition !**

Wigner transformation of the Kadanoff-Baym equation

- do Wigner transformation of the Kadanoff-Baym equation

$$F_{XP} = \int d^4(x-y) \;\; e^{iP_\mu(x^\mu - y^\mu)} \;\; F_{xy}$$

For any function F_{XY} with X=(x+y)/2 – space-time coordinate, P – 4-momentum

Convolution integrals convert under Wigner transformation as

$$\int d^4(x-y) \;\; e^{iP_\mu(x^\mu - y^\mu)} \;\; F_{1,xz} \odot F_{2,zy} \;=\; e^{-i\diamond} \; F_{1,PX} \; F_{2,PX}$$

Operator ◇ is a 4-dimentional generalizaton of the Poisson-bracket:

an infinite series in the differential operator ◇

$$\diamond\{\,F_1\,\}\{\,F_2\,\} \;:=\; \frac{1}{2}\left(\frac{\partial F_1}{\partial X_\mu}\,\frac{\partial F_2}{\partial P^\mu} \;-\; \frac{\partial F_1}{\partial P_\mu}\,\frac{\partial F_2}{\partial X^\mu}\right)$$

- consider only contribution up to first order in the gradients
  = a standard approximation of kinetic theory which is justified if the gradients in the mean spacial coordinate X are small

From Kadanoff-Baym equations to transport equations

➢ separate all retarded and advanced quantities – Geen functions and self- energies – into real and imaginary parts:

$$S^{ret,adv}_{XP} = Re S^{ret}_{XP} \mp \frac{i}{2} A_{XP} , \qquad \Sigma^{ret,adv}_{XP} = Re \Sigma^{ret}_{XP} \mp \frac{i}{2} \Gamma_{XP}$$

The imaginary part of the retarded propagator is given by the normalized spectral function A_{XP}:

$$A_{XP} = i \left[S^{ret}_{XP} - S^{adv}_{XP} \right] = -2\, Im\, S^{ret}_{XP}$$

$$\int \frac{dP_0^2}{4\pi} A_{XP} = 1$$

The imaginary part of the selfenergy corresponds to the width Γ_{XP};
then from Dyson-Schwinger equation:

$$Re S^{ret}_{XP} = \frac{P^2 - M_0^2 - Re\Sigma^{ret}_{XP}}{\Gamma_{XP}} A_{XP}$$

algebraic solution

The spectral function A_{XP} in first order gradient expansion (for bosons) :

$$A_{XP} = \frac{\Gamma_{XP}}{(P^2 - M_0^2 - Re\Sigma^{ret}_{XP})^2 + \Gamma^2_{XP}/4}$$

The real part of the retarded propagator in first order gradient expansion :

$$Re S^{ret}_{XP} = \frac{P^2 - M_0^2 - Re\Sigma^{ret}_{XP}}{(P^2 - M_0^2 - Re\Sigma^{ret}_{XP})^2 + \Gamma^2_{XP}/4}$$

A_{XP} and $Re\Sigma_{XP}^{ret}$ in first order gradient expansion depend ONLY on Σ_{XP}^{ret} !

From Kadanoff-Baym equations to generalized transport equations

After the first order gradient expansion of the Wigner transformed Kadanoff-Baym equations and separation into the real and imaginary parts one gets:

Generalized transport equations (GTE):

drift term — Vlasov term — backflow term — collision term = ‚gain' - ‚loss' term

$$\diamond \{ P^2 - M_0^2 - Re\Sigma_{XP}^{ret} \} \{ S_{XP}^{<} \} - \diamond \{ \Sigma_{XP}^{<} \} \{ Re S_{XP}^{ret} \} = \frac{i}{2} [\Sigma_{XP}^{>} S_{XP}^{<} - \Sigma_{XP}^{<} S_{XP}^{>}]$$

Backflow term incorporates the off-shell behavior in the particle propagation

! vanishes in the quasiparticle limit $A_{XP} \to \delta(p^2-M^2)$

- GTE: Propagation of the Green's function $iS^{<}_{XP}=A_{XP}N_{XP}$, which carries information not only on the number of particles (N_{XP}), but also on their properties, interactions and correlations (via A_{XP})

- Spectral function:

$$A_{XP} = \frac{\Gamma_{XP}}{(P^2 - M_0^2 - Re\Sigma_{XP}^{ret})^2 + \Gamma_{XP}^2/4}$$

$\Gamma_{XP} = -Im\,\Sigma_{XP}^{ret} = 2p_0\Gamma$ – ‚width' of spectral function

= reaction rate of particle (at space-time position X)

4-dimentional generalizaton of the Poisson-bracket:

$$\diamond \{ F_1 \} \{ F_2 \} := \frac{1}{2} \left(\frac{\partial F_1}{\partial X_\mu} \frac{\partial F_2}{\partial P^\mu} - \frac{\partial F_1}{\partial P_\mu} \frac{\partial F_2}{\partial X^\mu} \right)$$

- Life time $\tau = \dfrac{\hbar c}{\Gamma}$

W. Cassing , S. Juchem, NPA 665 (2000) 377; 672 (2000) 417; 677 (2000) 445

General testparticle off-shell equations of motion

W. Cassing , S. Juchem, NPA 665 (2000) 377; 672 (2000) 417; 677 (2000) 445

- Employ testparticle Ansatz for the real valued quantity $i\, S^<_{XP}$ -

$$F_{XP} = A_{XP} N_{XP} = i\, S^<_{XP} \sim \sum_{i=1}^{N} \delta^{(3)}(\vec{X} - \vec{X}_i(t))\ \delta^{(3)}(\vec{P} - \vec{P}_i(t))\ \delta(P_0 - \epsilon_i(t))$$

insert in generalized transport equations and determine equations of motion !

➔ General testparticle ‚Cassing off-shell equations of motion‘ for the time-like particles:

$$\frac{d\vec{X}_i}{dt} = \frac{1}{1 - C_{(i)}} \frac{1}{2\epsilon_i} \left[2\,\vec{P}_i + \vec{\nabla}_{P_i} Re\Sigma^{ret}_{(i)} + \frac{\epsilon_i^2 - \vec{P}_i^2 - M_0^2 - Re\Sigma^{ret}_{(i)}}{\Gamma_{(i)}} \vec{\nabla}_{P_i} \Gamma_{(i)} \right],$$

$$\frac{d\vec{P}_i}{dt} = -\frac{1}{1 - C_{(i)}} \frac{1}{2\epsilon_i} \left[\vec{\nabla}_{X_i} Re\Sigma^{ret}_i + \frac{\epsilon_i^2 - \vec{P}_i^2 - M_0^2 - Re\Sigma^{ret}_{(i)}}{\Gamma_{(i)}} \vec{\nabla}_{X_i} \Gamma_{(i)} \right],$$

$$\frac{d\epsilon_i}{dt} = \frac{1}{1 - C_{(i)}} \frac{1}{2\epsilon_i} \left[\frac{\partial Re\Sigma^{ret}_{(i)}}{\partial t} + \frac{\epsilon_i^2 - \vec{P}_i^2 - M_0^2 - Re\Sigma^{ret}_{(i)}}{\Gamma_{(i)}} \frac{\partial \Gamma_{(i)}}{\partial t} \right],$$

with $F_{(i)} \equiv F(t, \vec{X}_i(t), \vec{P}_i(t), \epsilon_i(t))$

$$C_{(i)} = \frac{1}{2\epsilon_i} \left[\frac{\partial}{\partial \epsilon_i} Re\Sigma^{ret}_{(i)} + \frac{\epsilon_i^2 - \vec{P}_i^2 - M_0^2 - Re\Sigma^{ret}_{(i)}}{\Gamma_{(i)}} \frac{\partial}{\partial \epsilon_i} \Gamma_{(i)} \right]$$

Note: the common factor $1/(1-C_{(i)})$ can be absorbed in an ‚eigentime‘ of particle (i) !

Limiting cases

- **$\Gamma(X,P) = \Gamma(X)$ - width depends only on space-time X:**

  $P = (P_0, \vec{P})$

  use M^2 as an independent variable $M^2 = P^2 - Re\Sigma^{ret}$

  and fix P_0 by $P_0^2 = \vec{P}^2 + M^2 + Re\Sigma^{ret}_{X\vec{P}M^2}$ **=>**

  follows:

$$\frac{dM_i^2}{dt} = \frac{M_i^2 - M_0^2}{\Gamma_{(i)}} \frac{d\Gamma_{(i)}}{dt}$$

i.e. the deviation of M_i^2 from the pole mass (squared) M_0^2 scales with Γ_i !

On-shell limit

❑ $\Gamma(X,P) \rightarrow 0$

$$A_{XP} = \frac{\Gamma_{XP}}{(P^2 - M_0^2 - Re\Sigma^{ret}_{XP})^2 + \Gamma^2_{XP}/4}$$

quasiparticle approximation :

A(X,P) = 2 p d(P^2-M^2)

||

Hamiltons equation of motion - independent on Γ !

$$\frac{d\vec{X}_i}{dt} = \frac{1}{1-C_{(i)}} \frac{1}{2\epsilon_i} \left[2\,\vec{P}_i + \vec{\nabla}_{P_i} Re\Sigma^{ret}_{(i)} + \xcancel{\frac{\epsilon_i^2 - \vec{P}_i^2 - M_0^2 - Re\Sigma^{ret}_{(i)}}{\Gamma_{(i)}} \vec{\nabla}_{P_i} \Gamma_{(i)}} \right],$$

$$\frac{d\vec{P}_i}{dt} = -\frac{1}{1-C_{(i)}} \frac{1}{2\epsilon_i} \left[\vec{\nabla}_{X_i} Re\Sigma^{ret}_i + \xcancel{\frac{\epsilon_i^2 - \vec{P}_i^2 - M_0^2 - Re\Sigma^{ret}_{(i)}}{\Gamma_{(i)}} \vec{\nabla}_{X_i} \Gamma_{(i)}} \right],$$

$$\frac{d\epsilon_i}{dt} = \frac{1}{1-C_{(i)}} \frac{1}{2\epsilon_i} \left[\frac{\partial Re\Sigma^{ret}_{(i)}}{\partial t} + \xcancel{\frac{\epsilon_i^2 - \vec{P}_i^2 - M_0^2 - Re\Sigma^{ret}_{(i)}}{\Gamma_{(i)}}} \frac{\partial \Gamma_{(i)}}{\partial t} \right],$$

Backflow term - which incorporates the off-shell behavior in the particle propagation - vanishes in the quasiparticle limit !

❑ **Γ(X,P) such that**

$\nabla_X \Gamma = 0$ and $\nabla_P \Gamma = 0$

E.g.: Γ = const

$\Gamma = \Gamma_{vacuum}$ (M)

<=>

'Vacuum' spectral function with constant or mass dependent width Γ: spectral function A_{XP} does NOT change the shape (and pole position) during propagation through the medium (backflow term vanishes also!)

⇒ Hamiltons equation of motion - independent on Γ !

Collision term in off-shell transport models

Collision term for reaction 1+2->3+4:

$$I_{coll}(X,\vec{P},M^2) = Tr_2 Tr_3 Tr_4 A(X,\vec{P},M^2) A(X,\vec{P}_2,M_2^2) A(X,\vec{P}_3,M_3^2) A(X,\vec{P}_4,M_4^2)$$

$$|G((\vec{P},M^2)+(\vec{P}_2,M_2^2)\rightarrow(\vec{P}_3,M_3^2)+(\vec{P}_4,M_4^2))|^2_{\mathcal{A},\mathcal{S}} \ \delta^{(4)}(P+P_2-P_3-P_4)$$

$$[\, N_{X\vec{P}_3M_3^2}\, N_{X\vec{P}_4M_4^2}\, \bar{f}_{X\vec{P}M^2}\, \bar{f}_{X\vec{P}_2M_2^2} \ - \ N_{X\vec{P}M^2}\, N_{X\vec{P}_2M_2^2}\, \bar{f}_{X\vec{P}_3M_3^2}\, \bar{f}_{X\vec{P}_4M_4^2}\,]$$

‚gain' term ‚loss' term

with $\bar{f}_{X\vec{P}M^2} = 1 + \eta\, N_{X\vec{P}M^2}$ and $\eta = \pm 1$ for bosons/fermions, respectively.

The trace over particles 2,3,4 reads explicitly

for fermions

$$Tr_2 = \sum_{\sigma_2,\tau_2} \frac{1}{(2\pi)^4} \int d^3P_2 \frac{dM_2^2}{2\sqrt{\vec{P}_2^2+M_2^2}}$$

for bosons

$$Tr_2 = \sum_{\sigma_2,\tau_2} \frac{1}{(2\pi)^4} \int d^3P_2 \frac{dP_{0,2}^2}{2}$$

additional integration

The transport approach and the particle spectral functions are fully determined once the in-medium transition amplitudes G are known in their off-shell dependence!

In-medium transition rates: G-matrix approach

Need to know in-medium transition amplitudes G and their off-shell dependence

$$|G((\vec{P}, M^2) + (\vec{P}_2, M_2^2) \rightarrow (\vec{P}_3, M_3^2) + (\vec{P}_4, M_4^2))|^2_{\mathcal{A},\mathcal{S}}$$

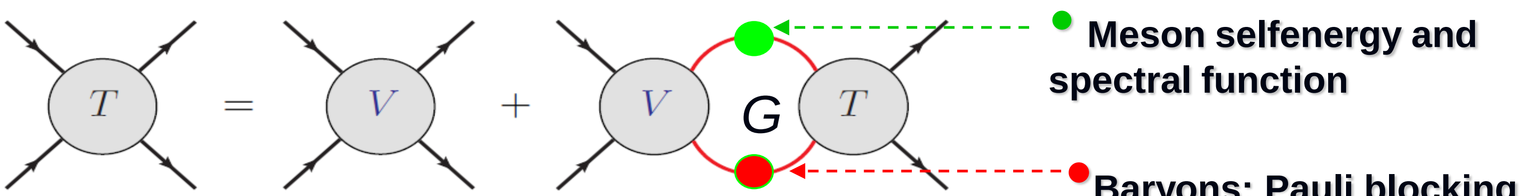
↓

Coupled channel G-matrix approach

Transition probability :

$$P_{1+2\rightarrow 3+4}(s) = \int d\cos(\theta)\ \frac{1}{(2s_1+1)(2s_2+1)} \sum_i \sum_\alpha\ G^\dagger G$$

with G(p,ρ,T) - G-matrix from the solution of coupled-channel equations:



- $\mathcal{T}_{ij}(\rho, T) = V_{ij} + V_{il}\ G_l(\rho, T)\ \mathcal{T}_{lj}(\rho, T)$

For strangeness:
D. Cabrera, L. Tolos, J. Aichelin, E.B., PRC C90 (2014) 055207; W. Cassing, L. Tolos, E.B., A. Ramos, NPA727 (2003) 59

Transition probabilities for $\pi Y \leftrightarrow K^- p$ $(Y = \Lambda, \Sigma)$

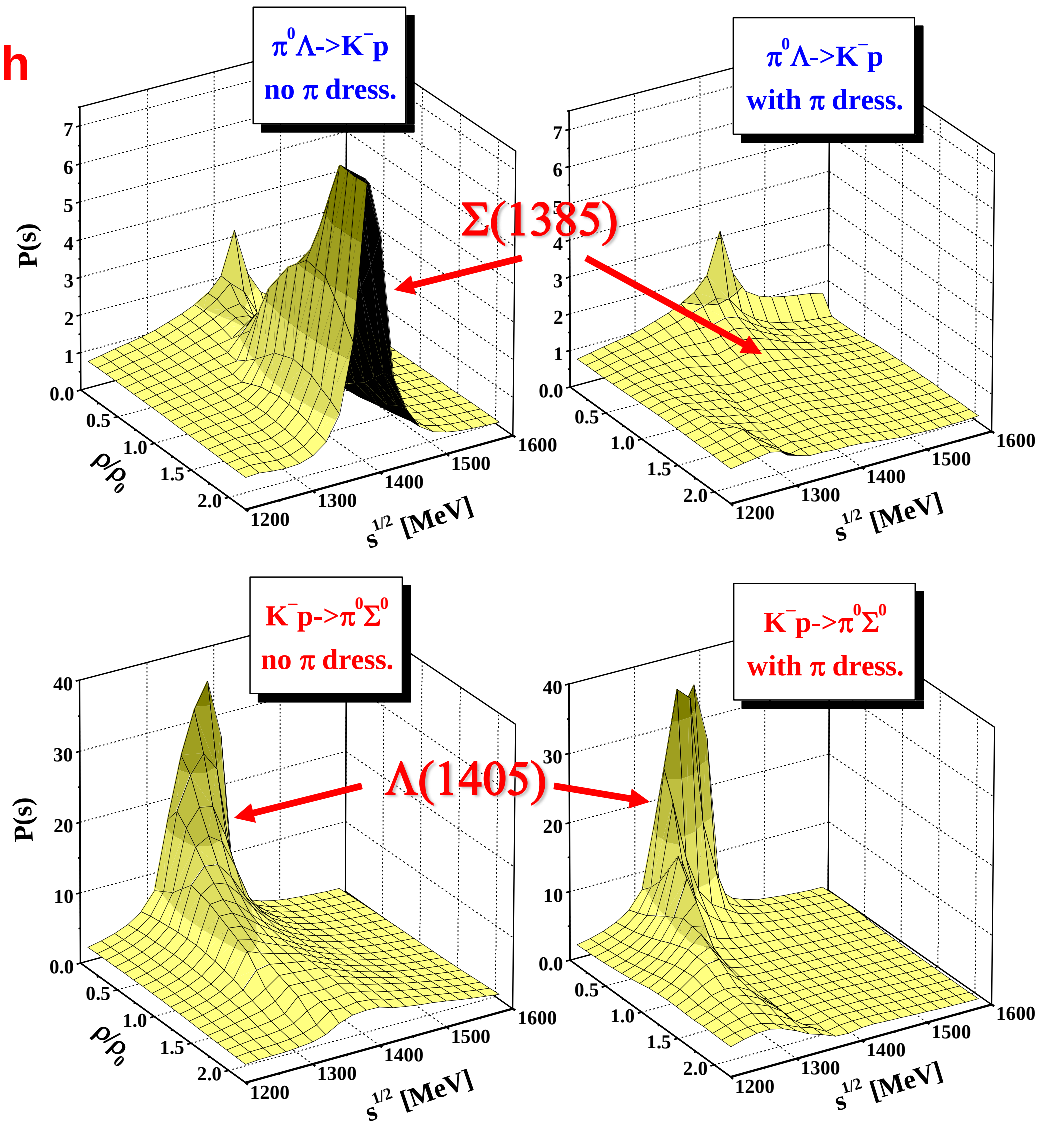
L. Tolos et al., NPA 690 (2001) 547

Coupled-channel G-matrix approach provides in-medium transition probabilities for different channels, e.g. $\pi Y \leftrightarrow K^- p$ $(Y = \Lambda, \Sigma)$

- **With pion dressing:**
  Λ(1405) and Σ(1385) melt away with baryon density

- **K^- absorption/production from πY collisions are strongly suppressed in the nuclear medium**

! πY is the dominant channel for K^- production in heavy-ion collisions !



W. Cassing, L. Tolos, E.L.B., A. Ramos, NPA 727 (2003) 59

Remarks on mean-field potential in off-shell transport models

- **Many-body theory:** Interacting relativistic particles have a **complex self-energy:**

$$\Sigma_{XP}^{ret} = Re\,\Sigma_{XP}^{ret} + i\,Im\,\Sigma_{XP}^{ret}$$

The neg. imaginary part $\Gamma_{XP} = -Im\,\Sigma_{XP}^{ret} = 2p_0\Gamma$ is related via $\Gamma = \Gamma_{coll} + \Gamma_{dec}$ to the inverse livetime of the particle $\tau \sim 1/\Gamma$.

- The **collision width** Γ_{coll} is determined from the **loss term** of the collision integral I_{coll}

$$-I_{coll}(loss) = \Gamma_{coll}(X,\vec{P},M^2)N_{X\vec{P}M^2}$$

- By **dispersion relation** (Kramers–Kronig relation) we get a contribution to the **real part of self-energy**:

$$Re\,\Sigma_{XP}^{ret}(p_0) = P\int_0^\infty dq\,\frac{Im\,\Sigma_{XP}^{ret}(q)}{(q-p_0)}$$

which gives **a mean-field potential U_{XP}** via: $Re\,\Sigma_{XP}^{ret}(p_0) = 2p_0 U_{XP}$

➔ The **complex self-energy** relates in a self-consistent way to the self-generated mean-field potential and collision width (inverse lifetime)

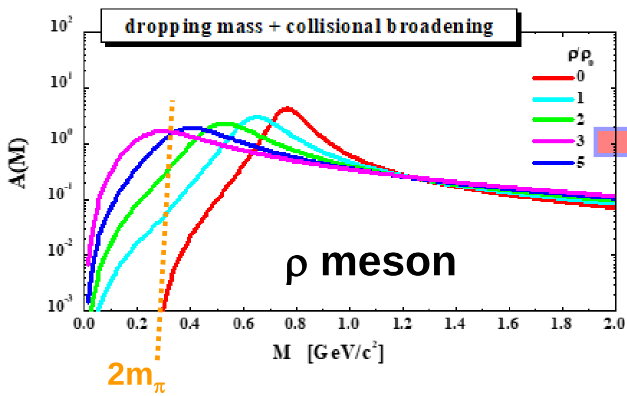
In-medium
$\rho >> \rho_0$

Off-shell vs. on-shell transport dynamics

Time evolution of the mass distribution of ρ and ω mesons for central C+C collisions (b=1 fm) at 2 A GeV for dropping mass + collisional broadening scenario

$$A(M,p,\rho) = \frac{2}{\pi}\frac{M^2\Gamma_{tot}(M,p,\rho)}{(M^2 - M_0^2 - Re\Sigma^{ret}) + \left(M\Gamma_{tot}(M,p,\rho)\right)^2},$$

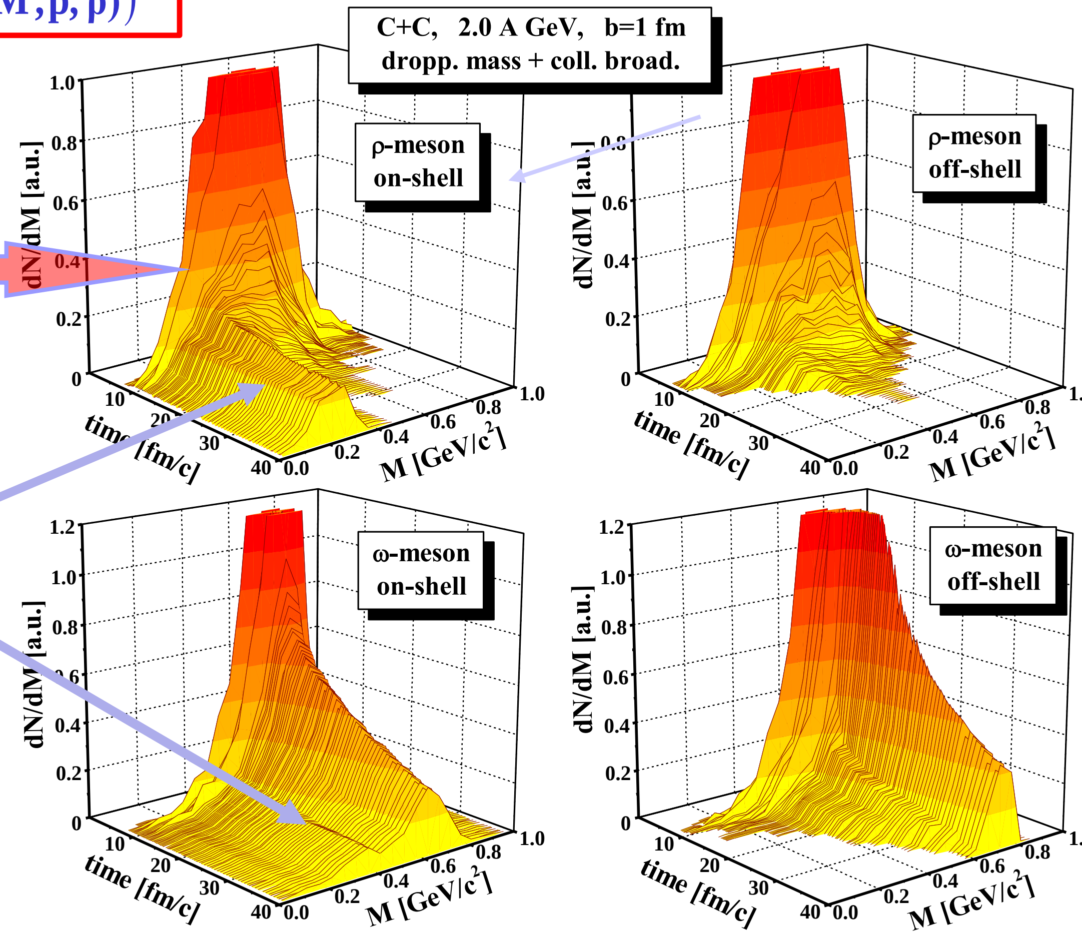
width $\Gamma \sim -Im\,\Sigma^{ret}/M$



On-shell BUU:
low mass ρ and ω mesons live forever (and shine ‚fake' dileptons)!

The off-shell spectral function becomes on-shell in the vacuum dynamically by propagation through the medium!

On-shell **Off-shell**



E.L.B. &W. Cassing, NPA 807 (2008) 214

Detailed balance on the level of 2←→n: treatment of multi-particle collisions in transport approaches

W. Cassing, NPA 700 (2002) 618

Generalized collision integral for *n* ←→ *m* reactions:

$$I_{coll} = \sum_n \sum_m I_{coll}[n \leftrightarrow m]$$

$$I^i_{coll}[n \leftrightarrow m] =$$
$$\frac{1}{2} N^m_n \sum_\nu \sum_\lambda \left(\frac{1}{(2\pi)^4}\right)^{n+m-1} \int \left(\prod_{j=2}^{n} d^4p_j \; A_j(x,p_j)\right) \left(\prod_{k=1}^{m} d^4p_k \; A_k(x,p_k)\right)$$
$$\times A_i(x,p) \; W_{n,m}(p,p_j;i,\nu \mid p_k;\lambda) \; (2\pi)^4 \; \delta^4(p^\mu + \sum_{j=2}^{n} p^\mu_j - \sum_{k=1}^{m} p^\mu_k)$$
$$\times [\tilde{f}_i(x,p) \prod_{k=1}^{m} f_k(x,p_k) \prod_{j=2}^{n} \tilde{f}_j(x,p_j) - f_i(x,p) \prod_{j=2}^{n} f_j(x,p_j) \prod_{k=1}^{m} \tilde{f}_k(x,p_k)].$$

$\tilde{f} = 1 + \eta f$ **is Pauli-blocking or Bose-enhancement factors;**
η=1 for bosons and η=-1 for fermions

$W_{n,m}(p,p_j;i,\nu \mid p_k;\lambda)$ **is a transition probability**

Antibaryon production in heavy-ion reactions

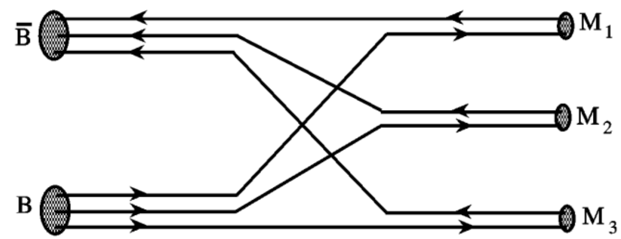
W. Cassing, NPA 700 (2002) 618
E. Seifert, W. Cassing, 1710.00665, 1801.07557

Multi-meson fusion reactions

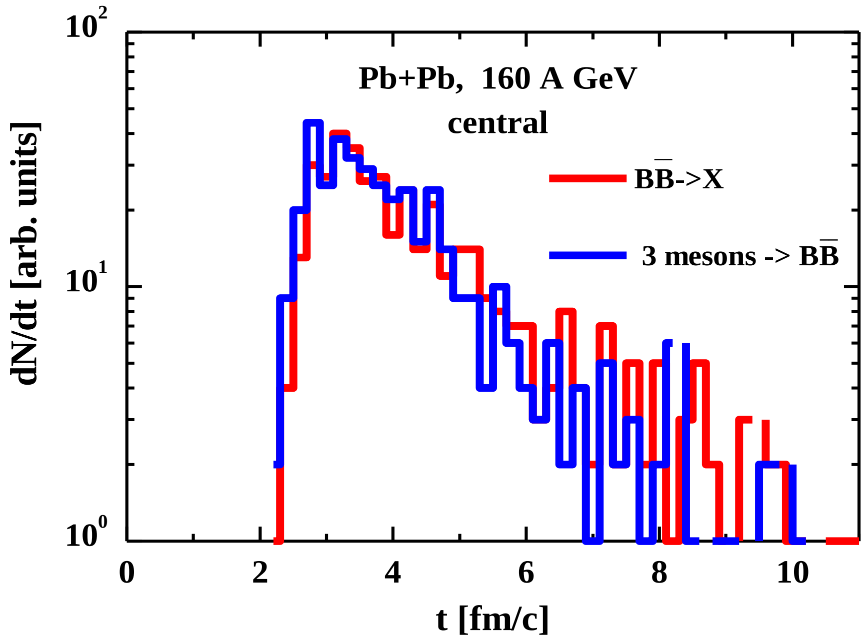
$m_1+m_2+...+m_n \leftrightarrow B+\bar{B}$

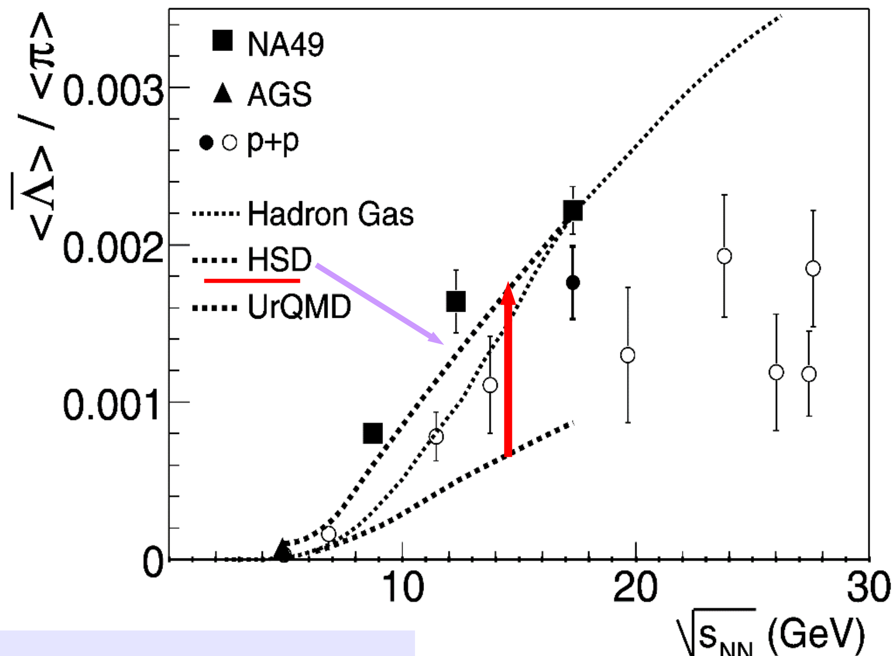
$m=\pi,\rho,\omega,..$ $B=p,\Lambda,\Sigma,\Omega$, (>2000 channels)

- important for anti-proton, anti-lambda, anti-Xi, anti-Omega dynamics !



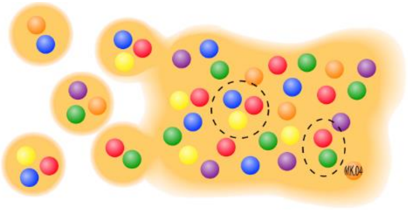
2←→3





→ approximate equilibrium of annihilation and recreation

Goal: microscopic transport description of the partonic and hadronic phase



Problems:

- ❑ How to model a QGP phase in line with lQCD data?
- ❑ How to solve the hadronization problem?

Ways to go:

pQCD based models:

- QGP phase: pQCD cascade
- hadronization: quark coalescence

➔ AMPT, HIJING, BAMPS

‚Hybrid' models:

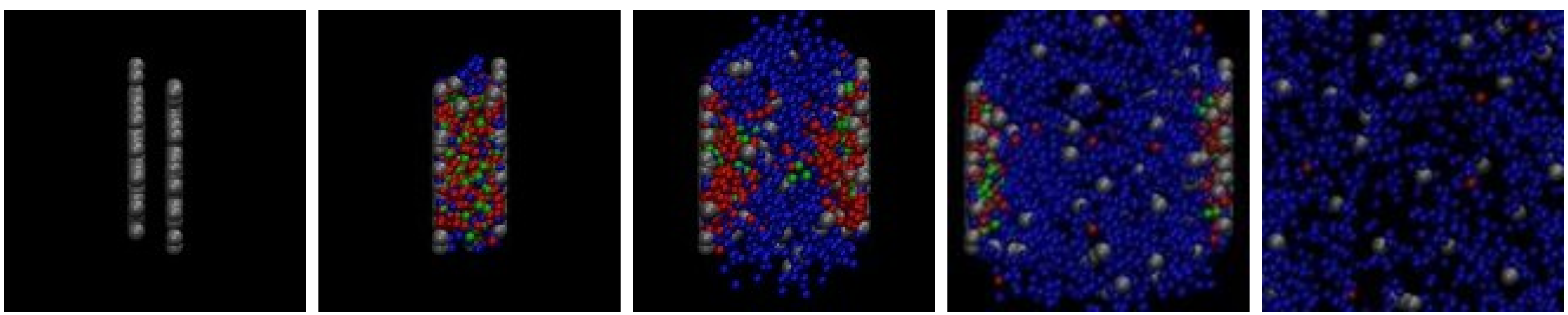
- QGP phase: hydro with QGP EoS
- hadronic freeze-out: after burner - hadron-string transport model

➔ Hybrid-UrQMD

- microscopic transport description of the partonic and hadronic phase in terms of strongly interacting dynamical quasi-particles and off-shell hadrons

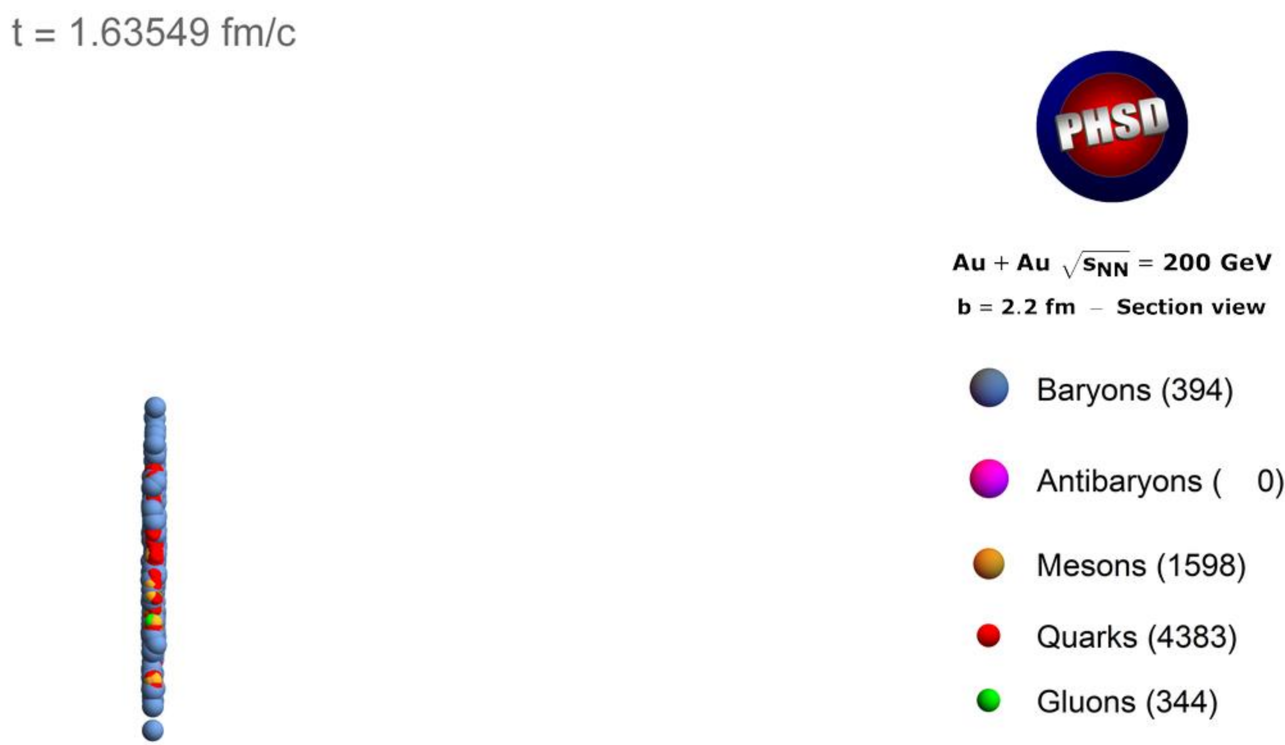
➔ PHSD

‚Bulk‘ properties in Au+Au



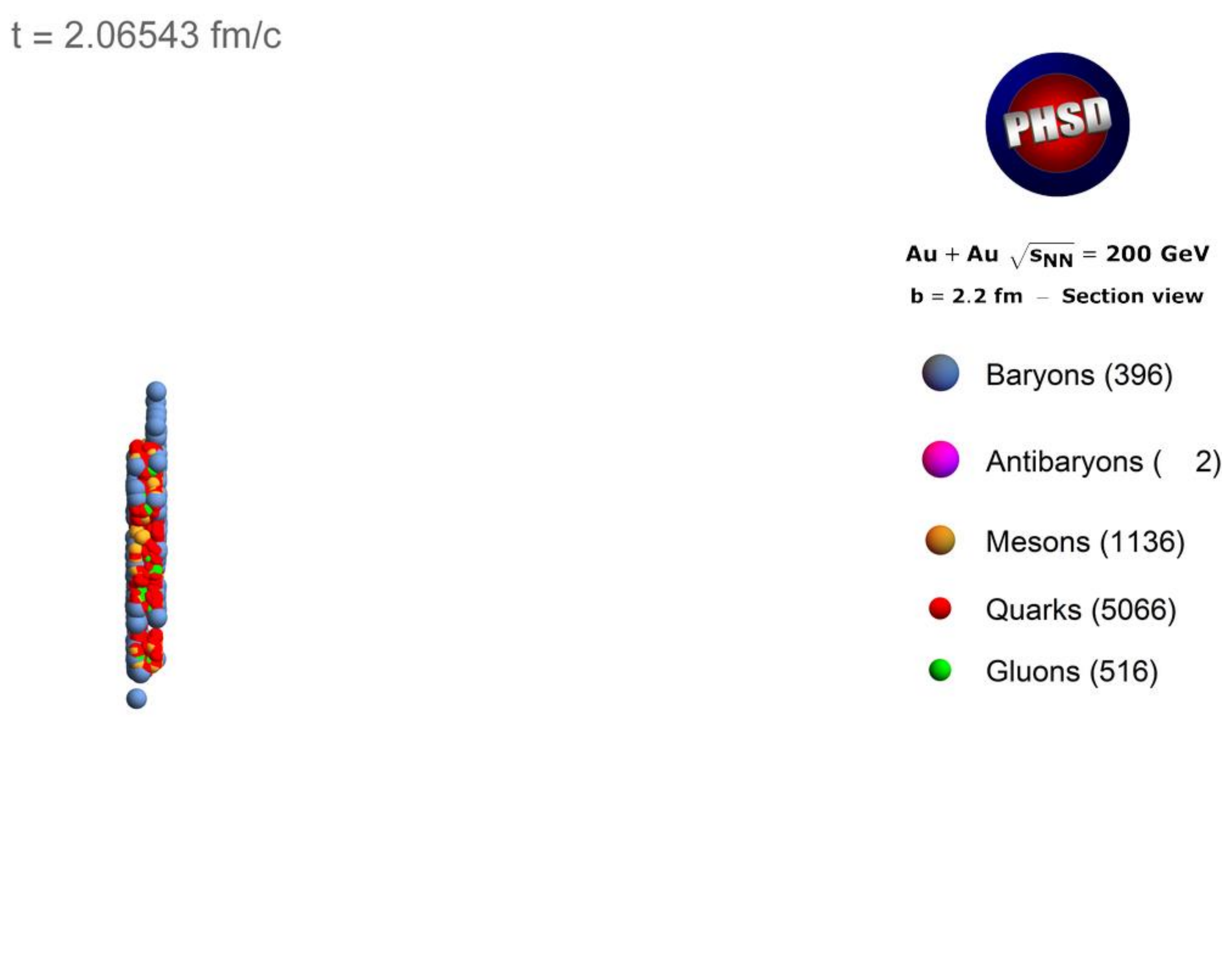
Au+Au at 200 A GeV, b=2.2 fm

Au+Au at 200 A GeV, b=2.2 fm



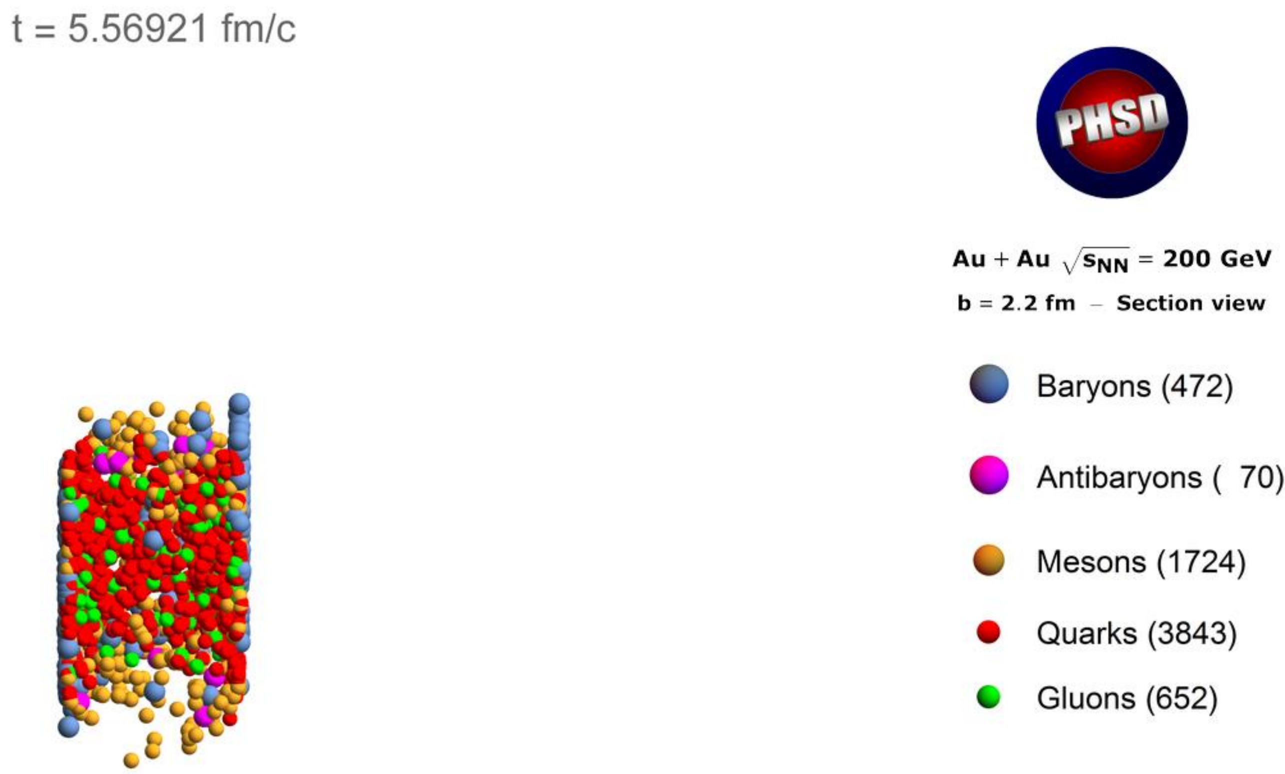
P.Moreau

Au+Au at 200 A GeV, b=2.2 fm



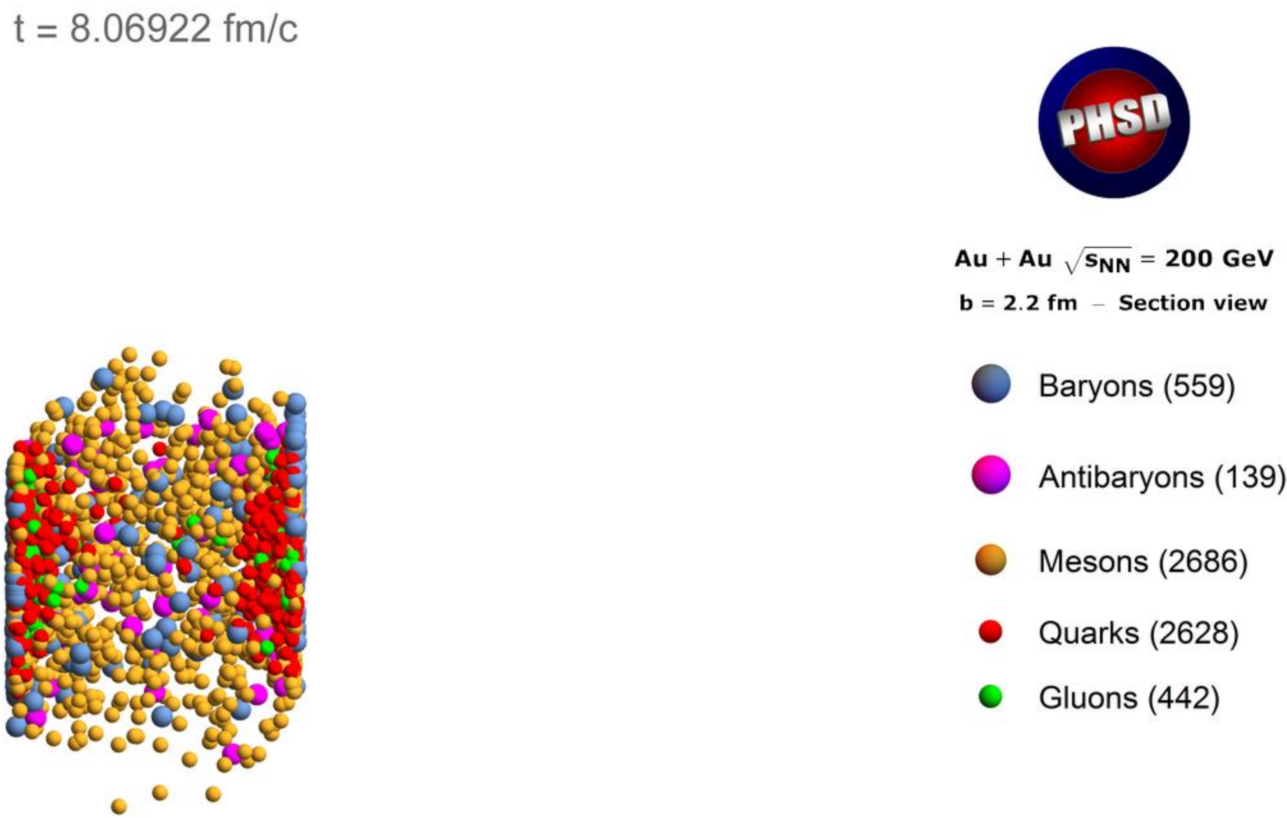
Au+Au at 200 A GeV, b=2.2 fm

Au+Au at 200 A GeV, b=2.2 fm



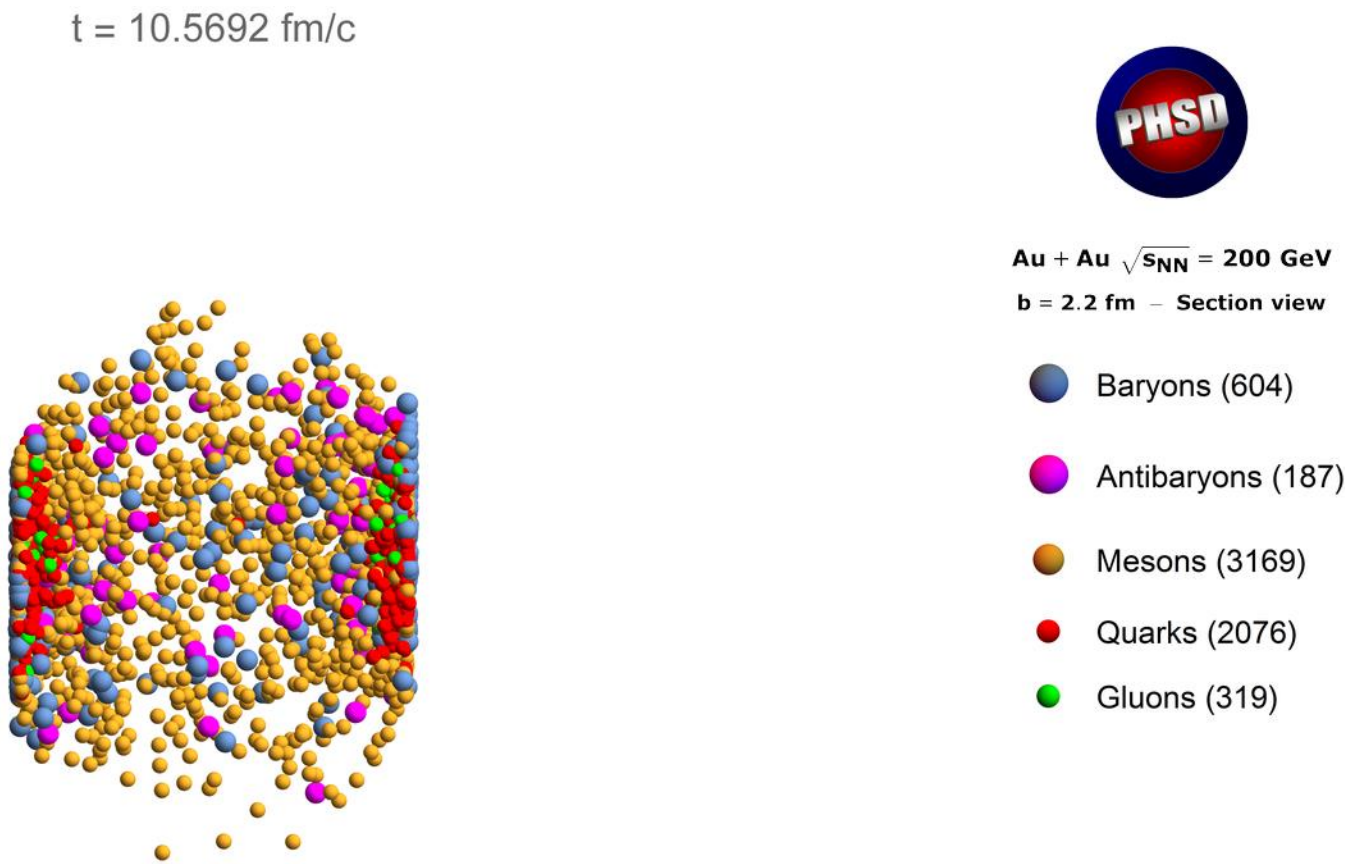
P.Moreau

Au+Au at 200 A GeV, b=2.2 fm



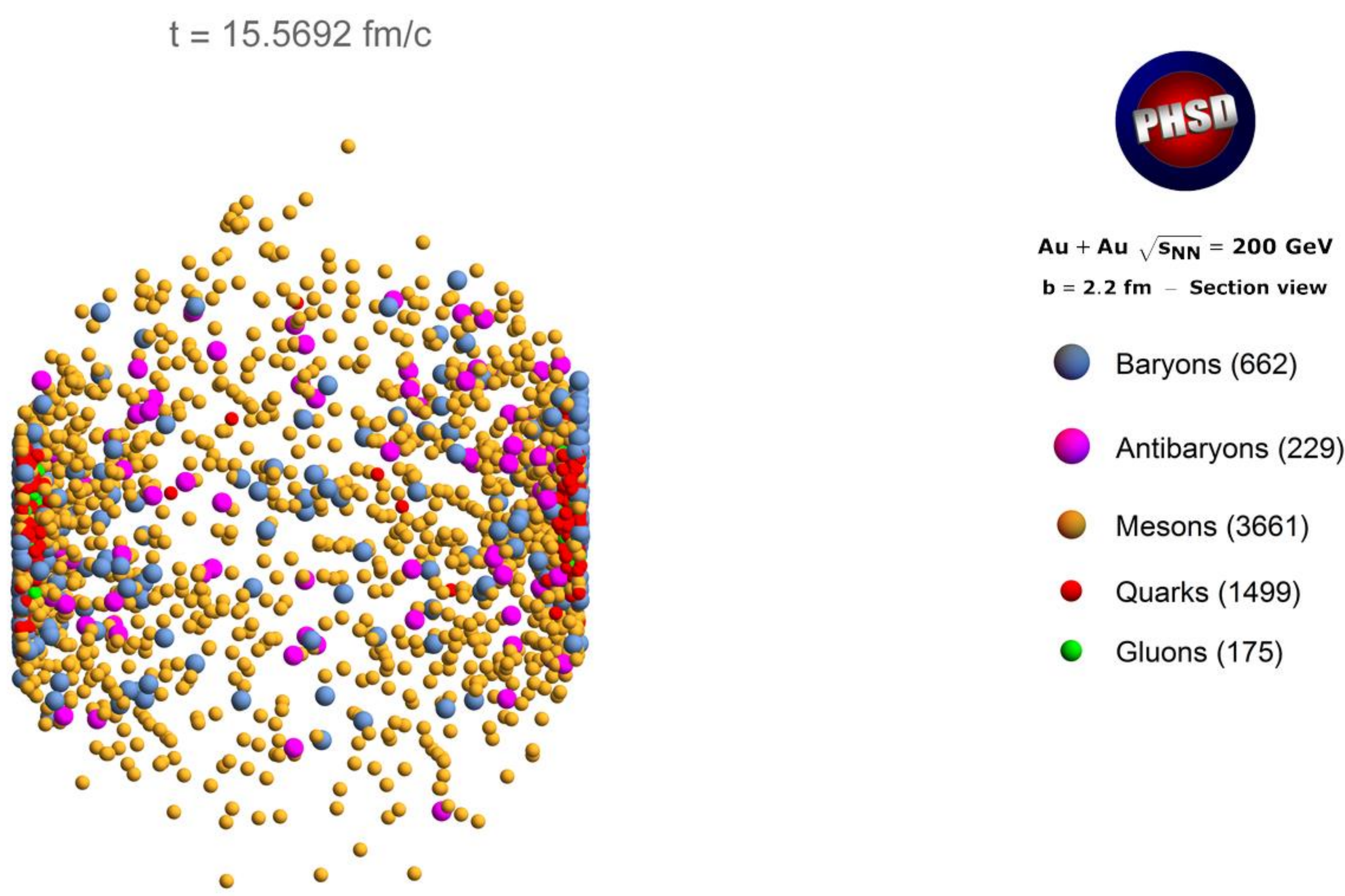
P.Moreau

Au+Au at 200 A GeV, b=2.2 fm



P.Moreau

Au+Au at 200 A GeV, b=2.2 fm



P.Moreau

Au+Au at 200 A GeV, b=2.2 fm

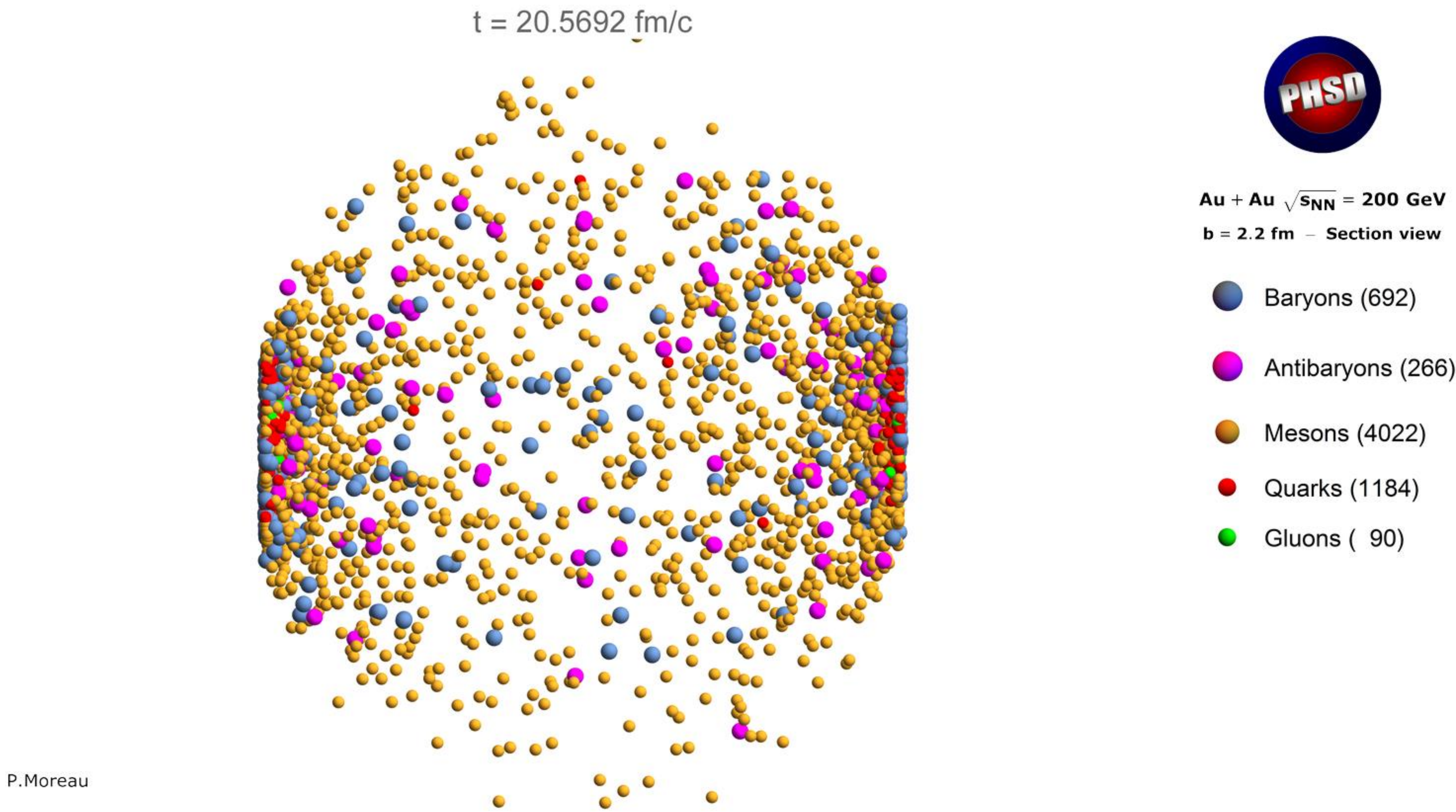
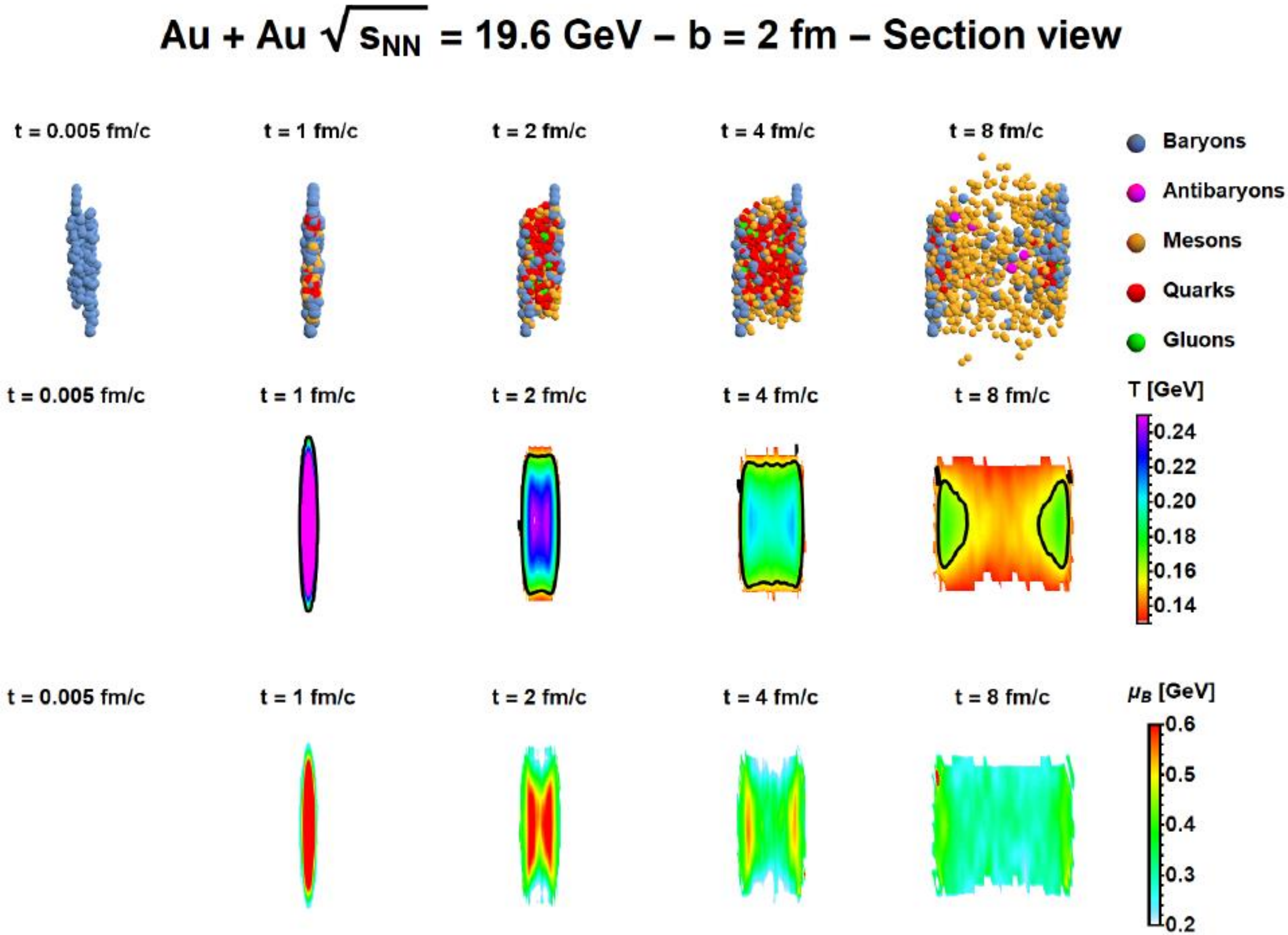
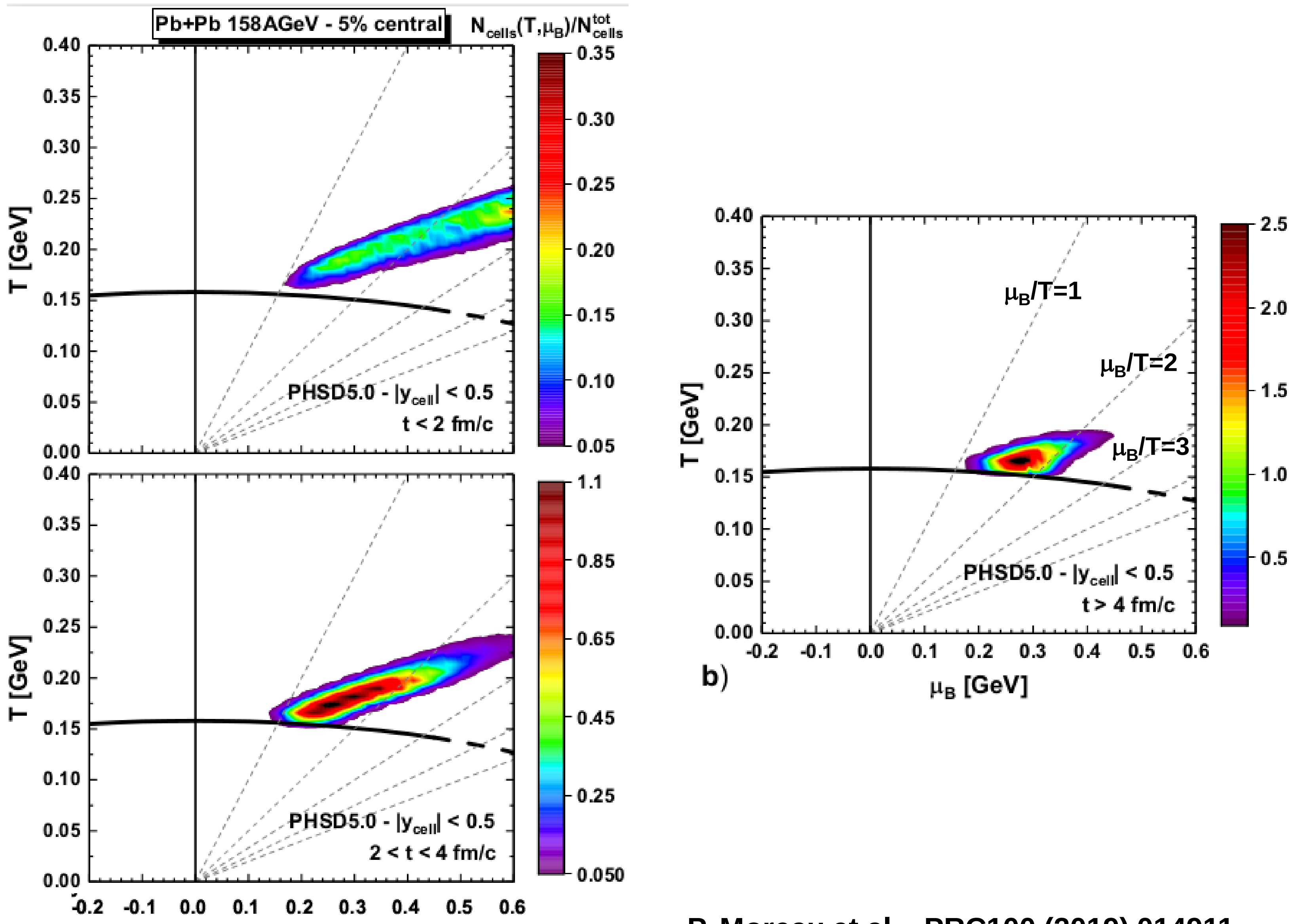


Illustration for a HIC ($\sqrt{s_{NN}} = 19.6$ GeV)



P. Moreau

Illustration for HIC ($\sqrt{s_{NN}} = 17$ GeV)

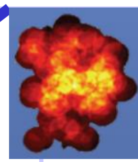


P. Moreau et al., PRC100 (2019) 014911

Dynamical models for HIC

Macroscopic

hydro-models:
- description of QGP and hadronic phase by hydrodanamical equations for fluid
- assumption of local equilibrium
- EoS with phase transition from QGP to HG
- initial conditions (e-b-e, fluctuating)



ideal
(Jyväskylä,SHASTA, TAMU, …)

viscous
(Romachkke,(2+1)D VISH2+1, (3+1)D MUSIC,…)

fireball models:
- no explicit dynamics: parametrized time evolution (TAMU)

Microscopic

Non-equilibrium microscopic transport models – based on many-body theory

Hadron-string models
(UrQMD, IQMD, HSD, QGSM, SMASH …)

Partonic cascades
pQCD based
(Duke, BAMPS, …)

Parton-hadron models:
- QGP: pQCD based cascade
- massless q, g
- hadronization: coalescence

(AMPT, HIJING)

- QGP: lQCD EoS
- massive quasi-particles (q and g with spectral functions) in self-generated mean-field
- dynamical hadronization
- HG: off-shell dynamics

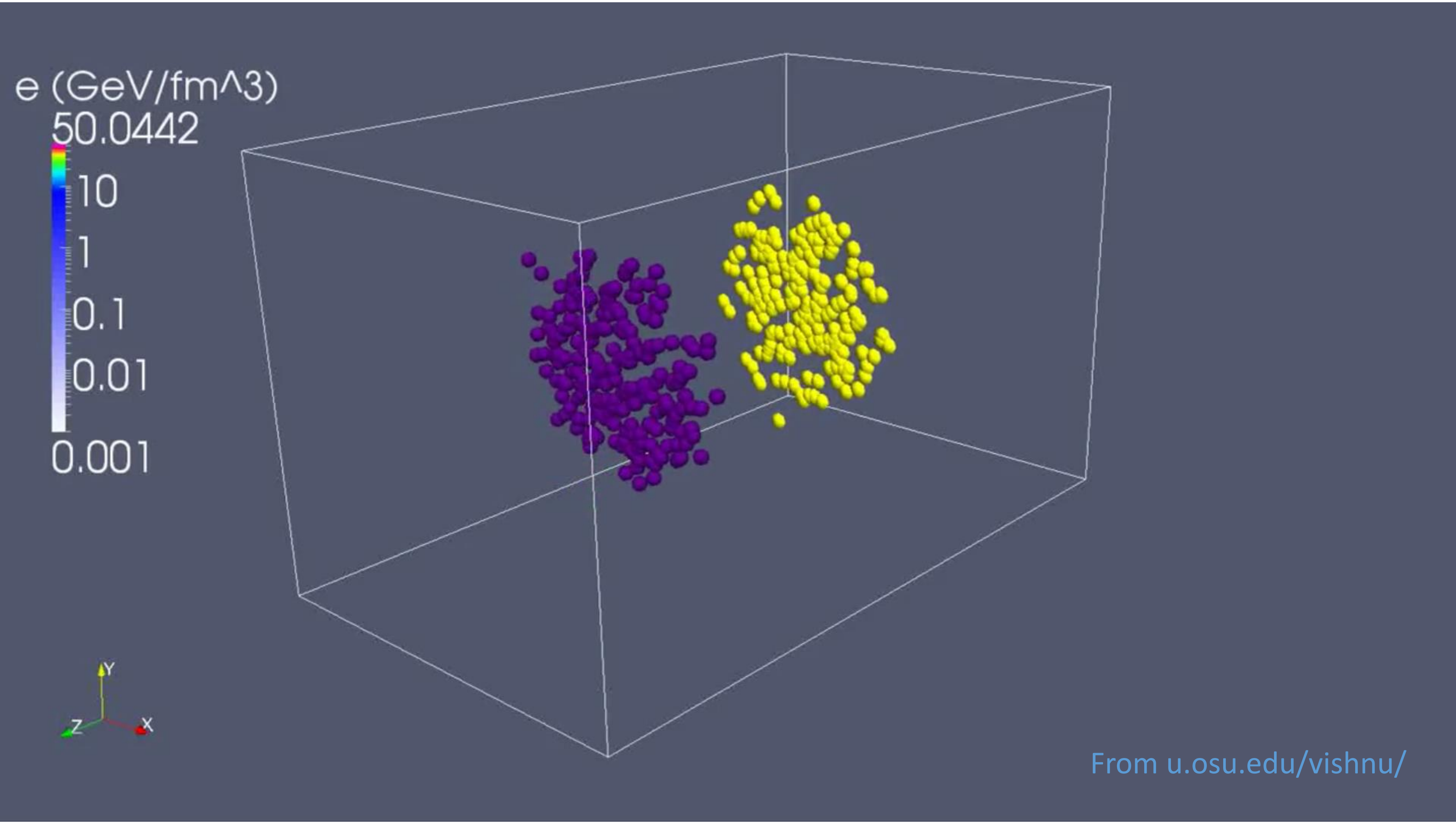
(applicable for strongly interacting systems)

'Hybrid'

QGP phase: hydro with QGP EoS
- hadronic freeze-out: after burner - hadron-string transport model

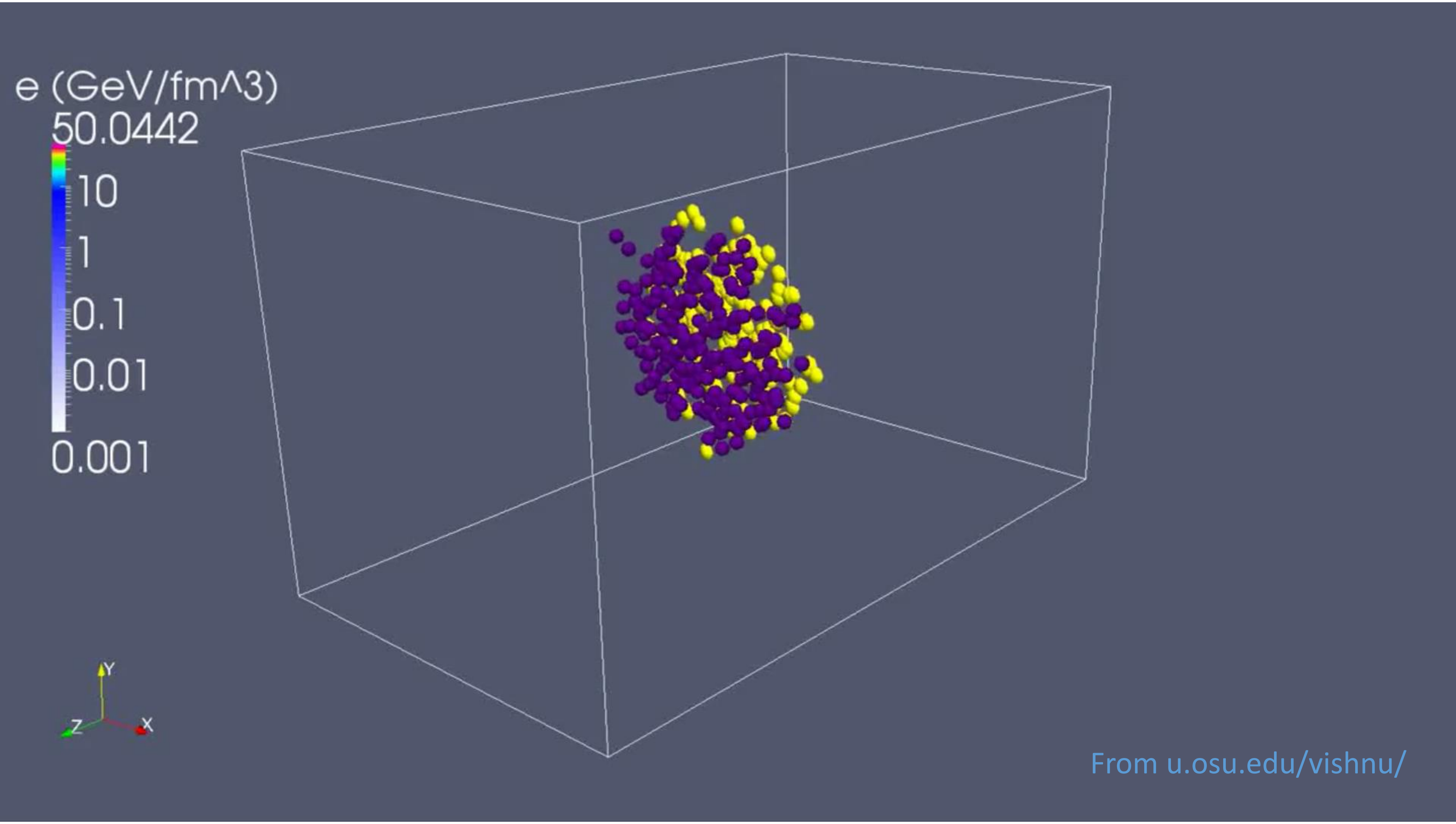
(‚hybrid'-UrQMD, EPOS, …)

Stages of a collision in VISHNU



From u.osu.edu/vishnu/

Stages of a collision in VISHNU



From u.osu.edu/vishnu/

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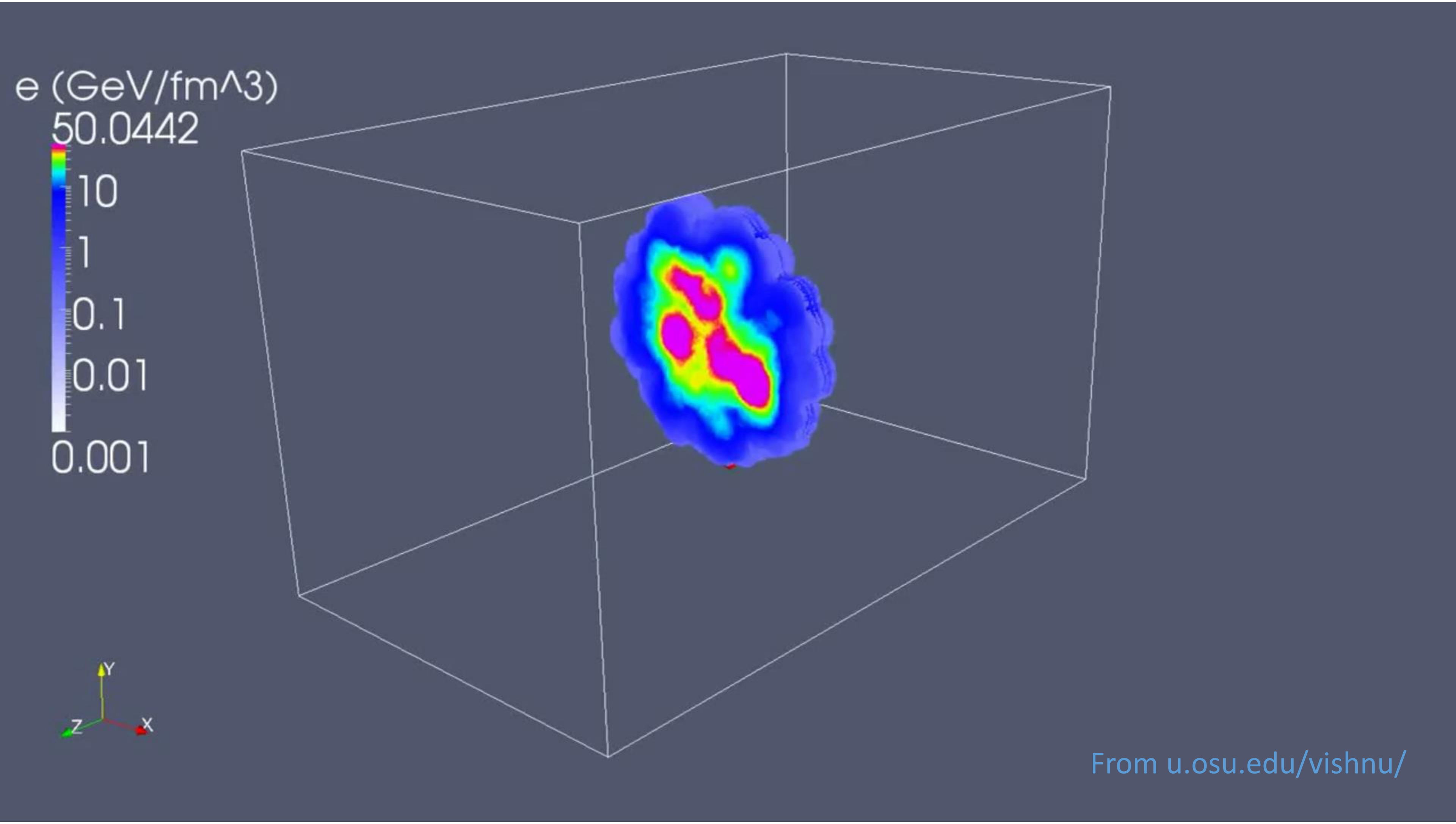

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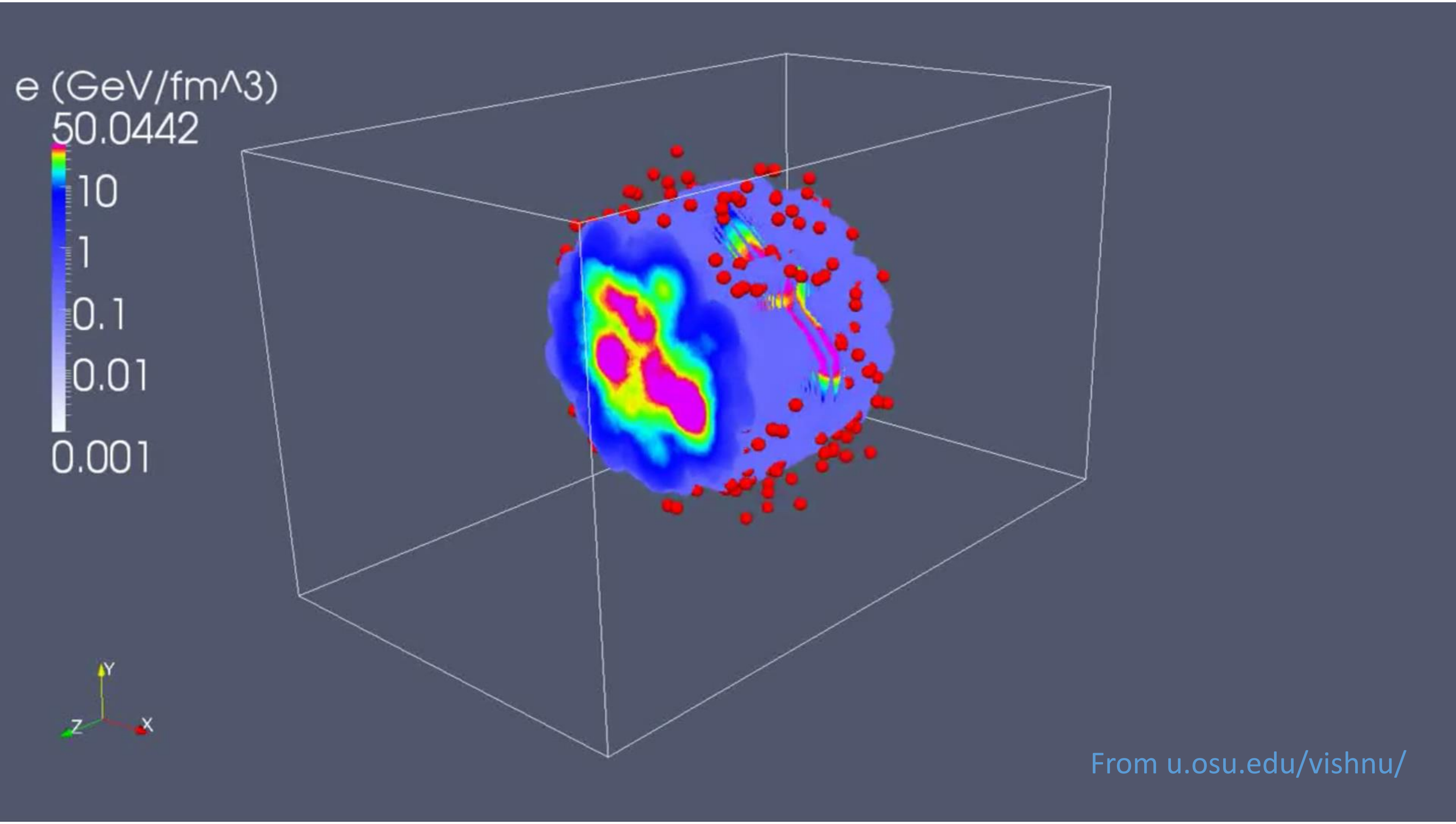

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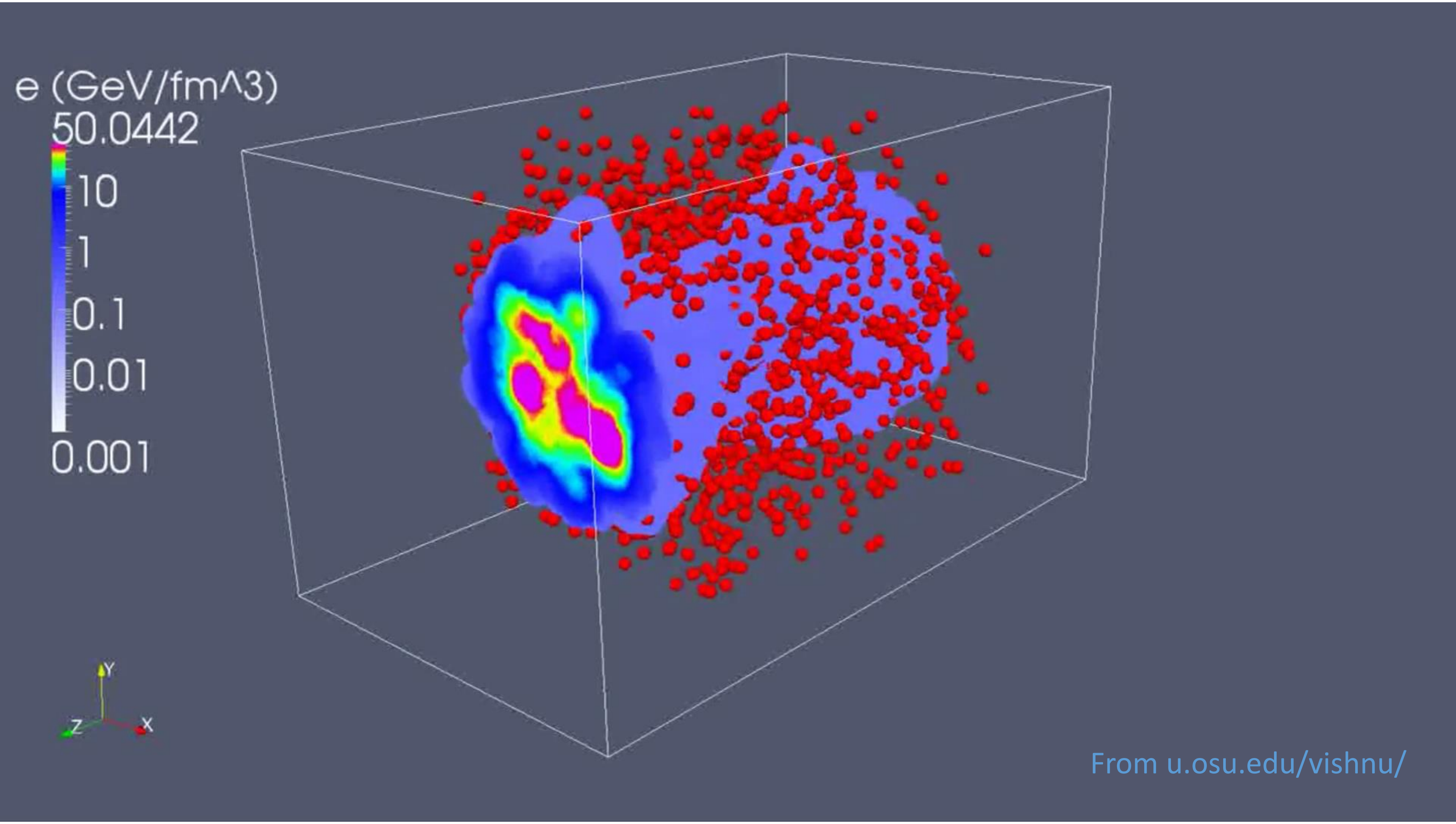

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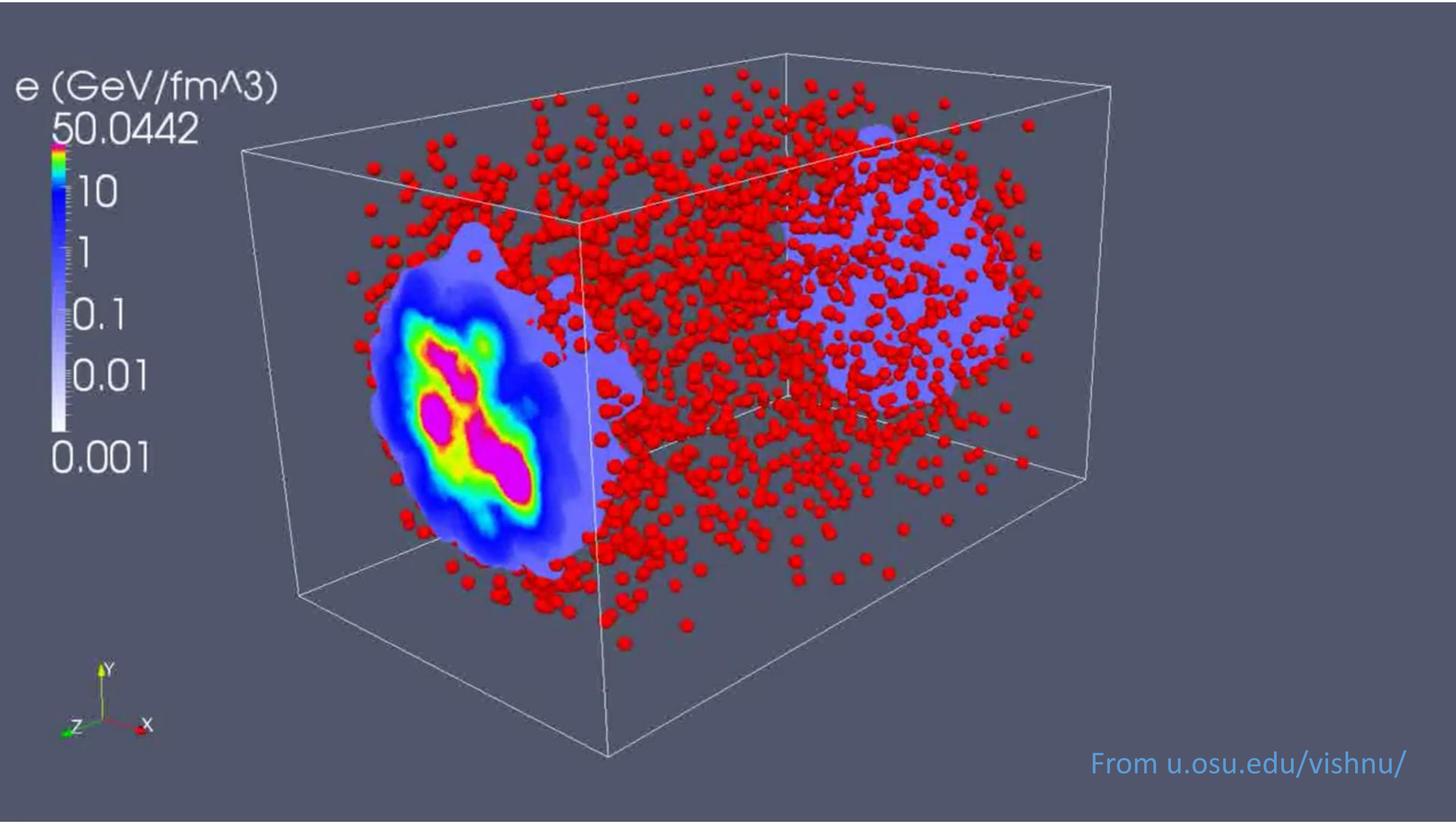
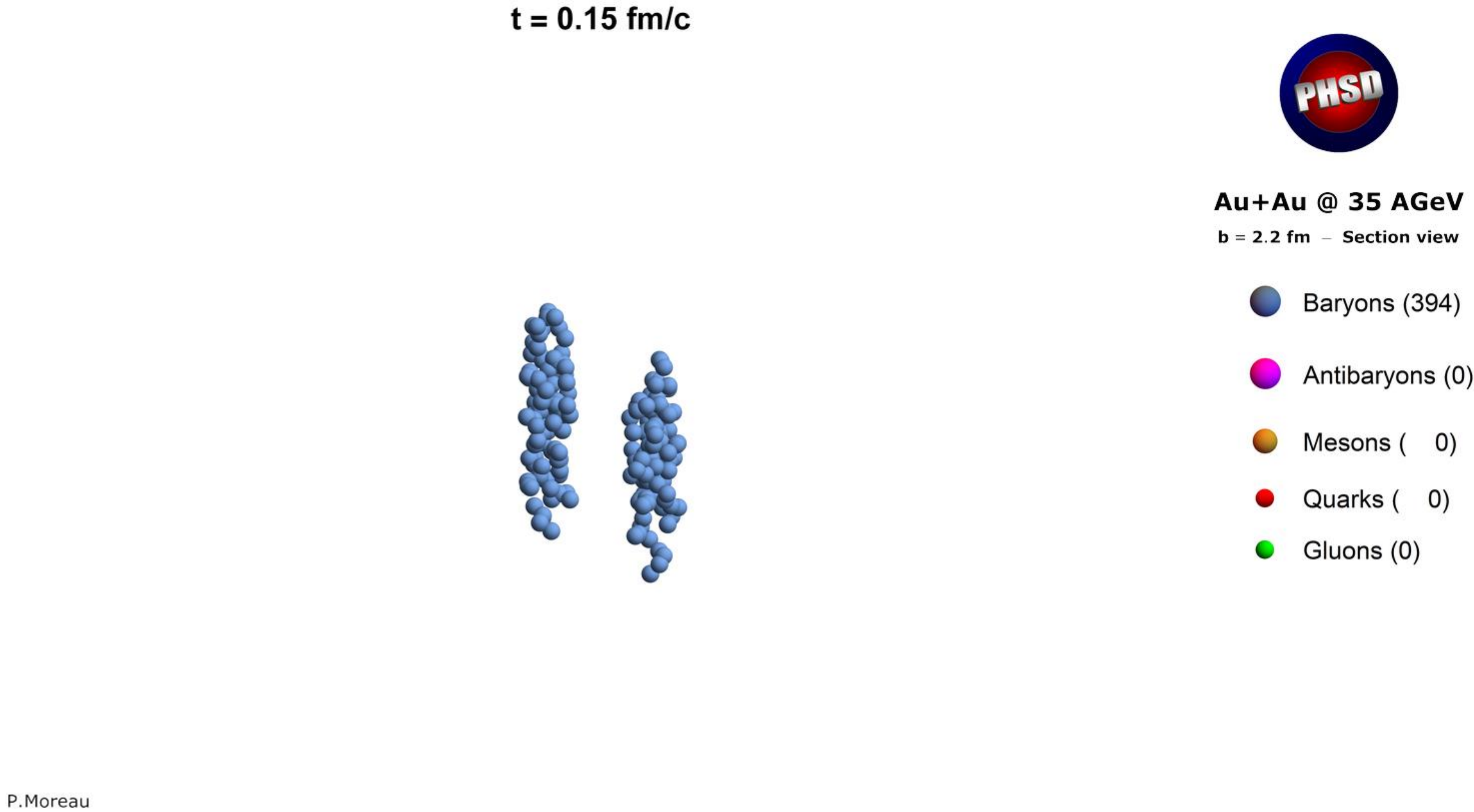
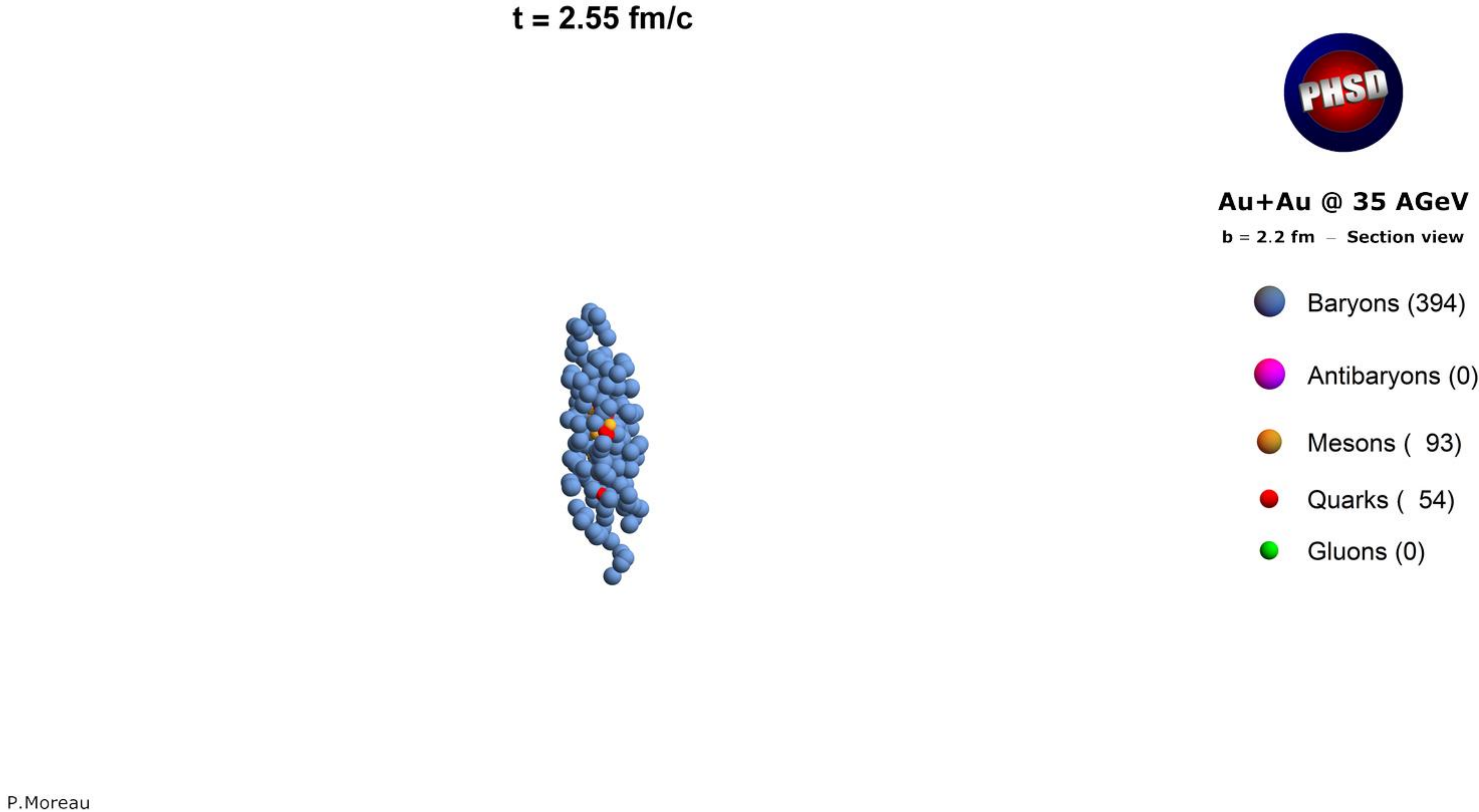


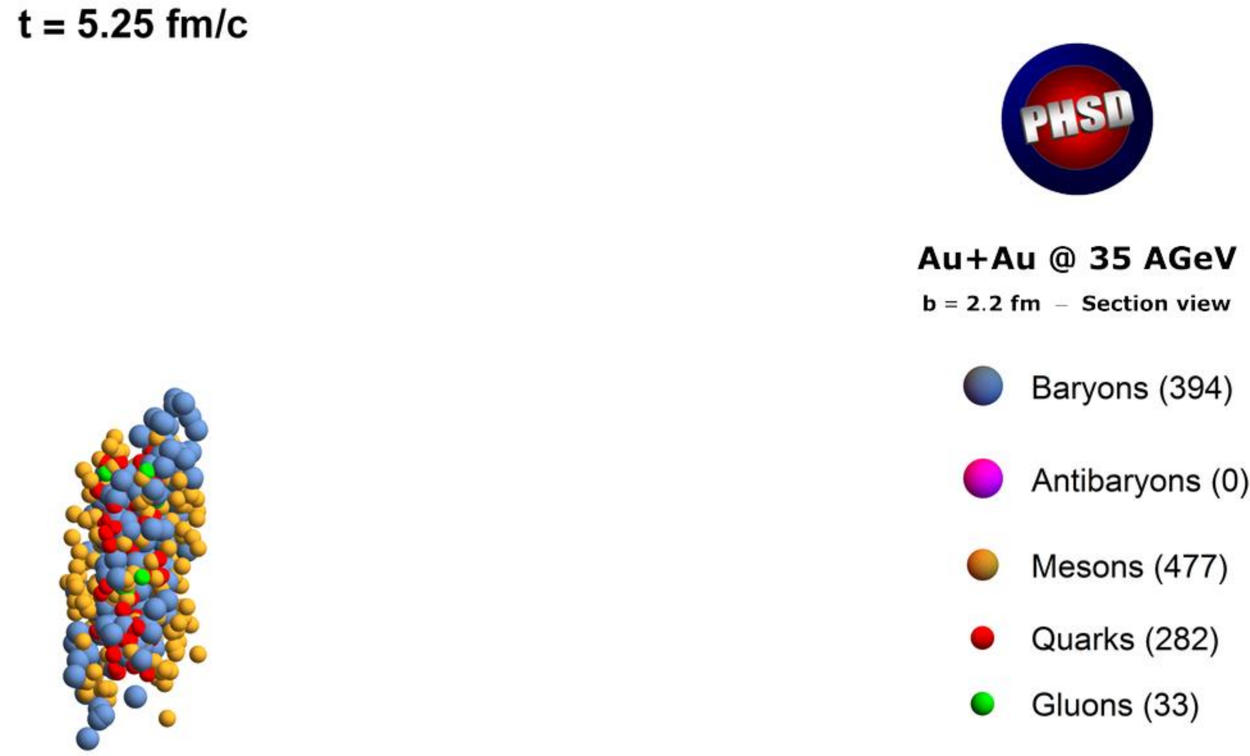
Stages of a collision in PHSD



Stages of a collision in PHSD

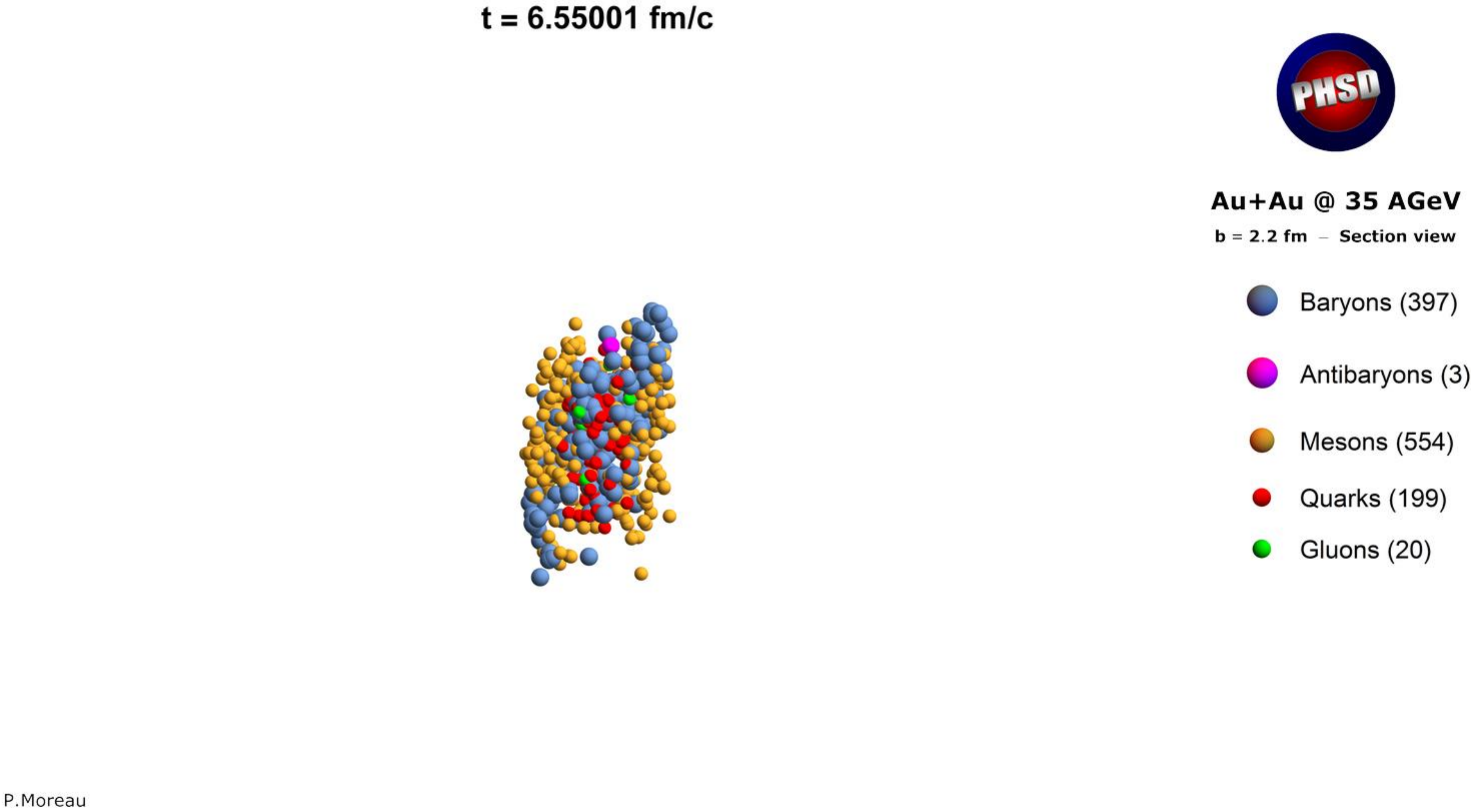


Stages of a collision in PHSD



P.Moreau

Stages of a collision in PHSD



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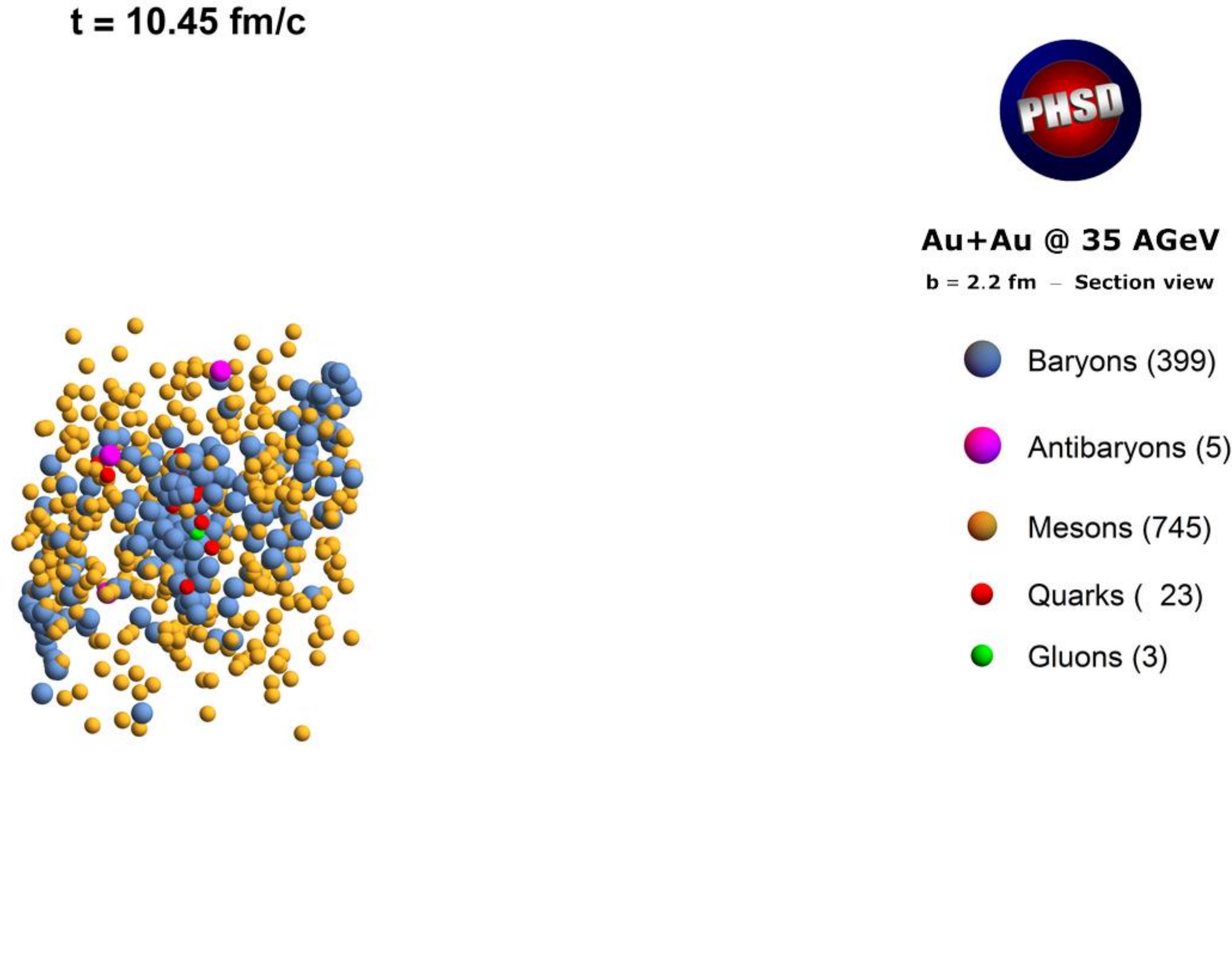

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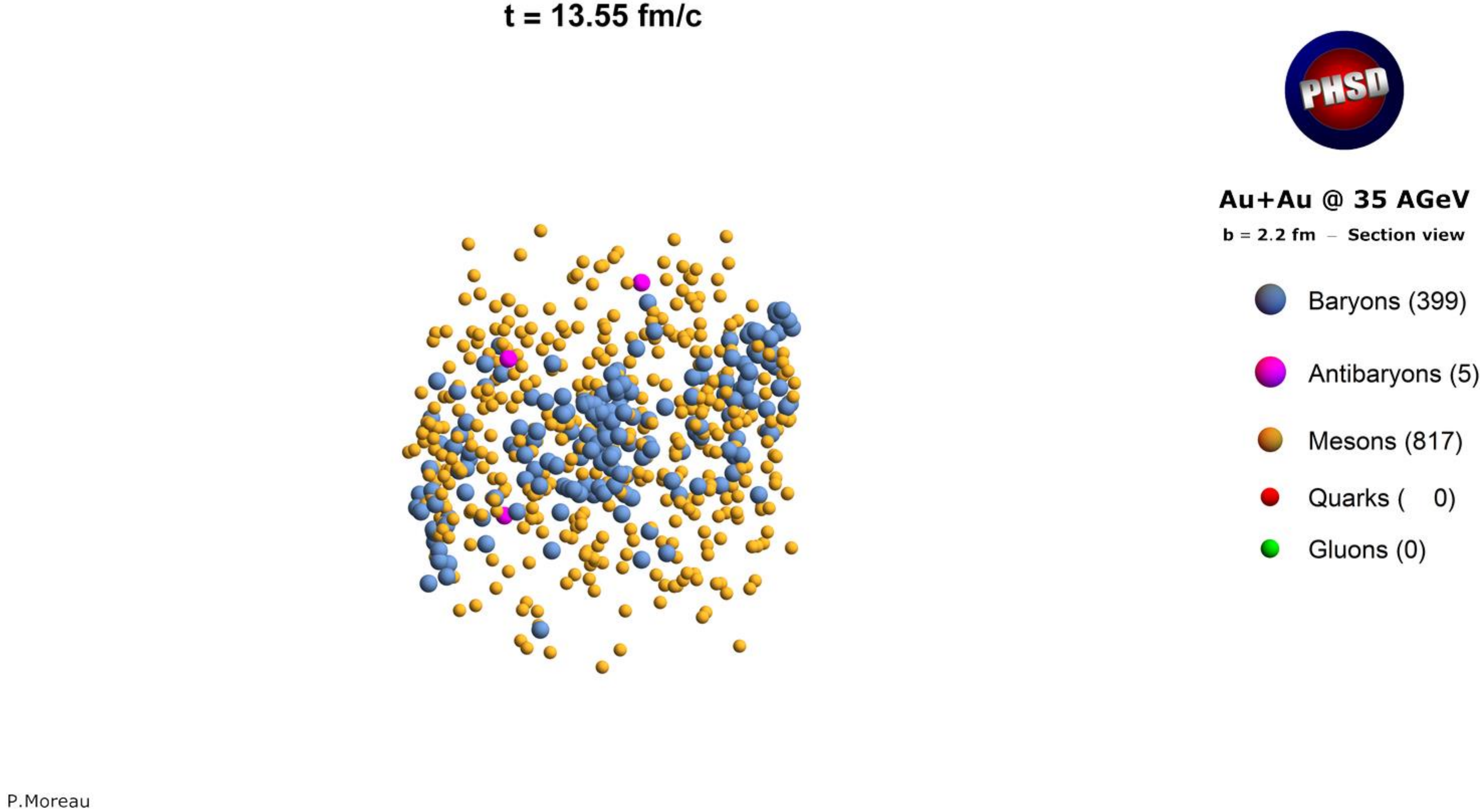

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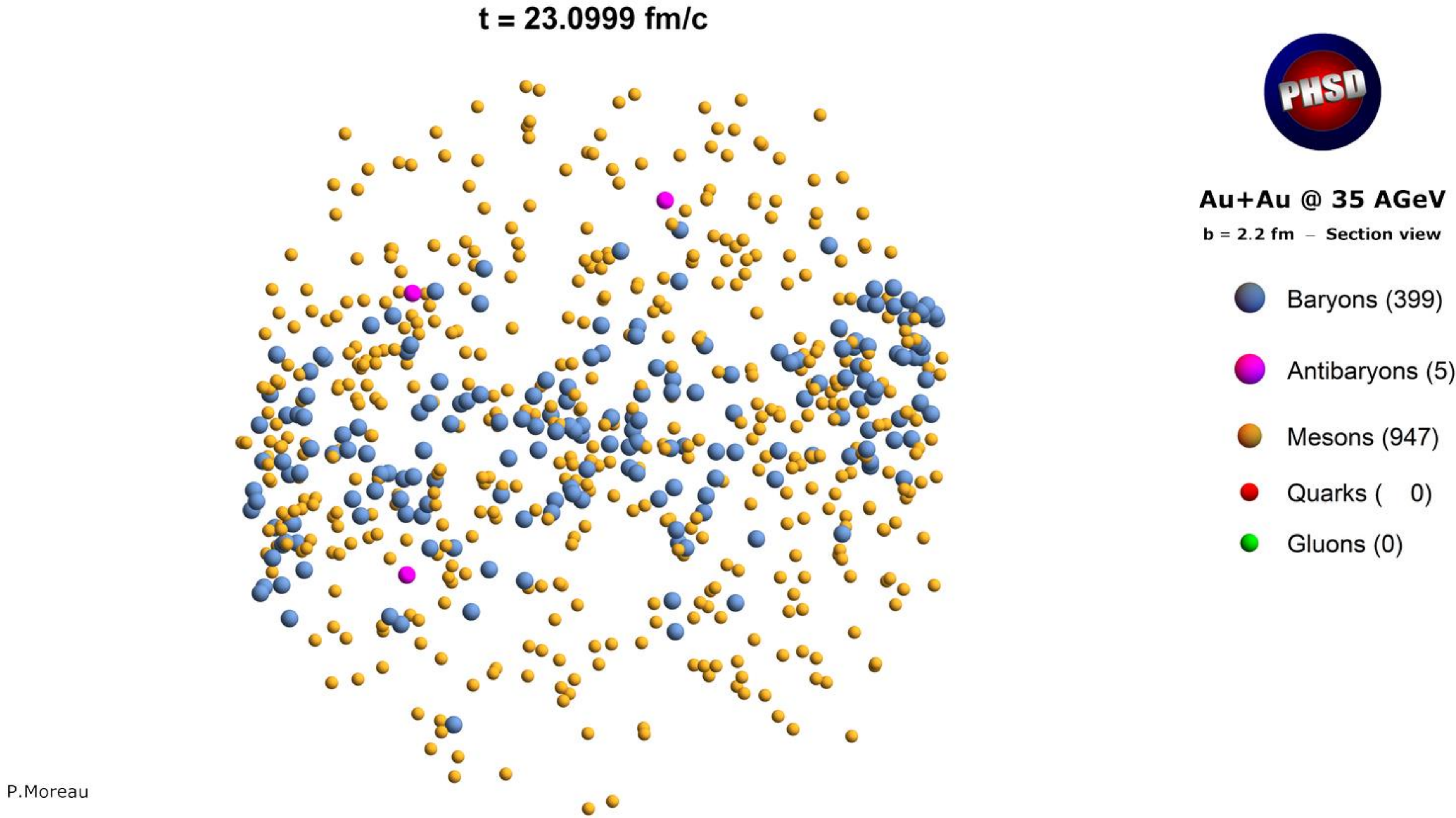

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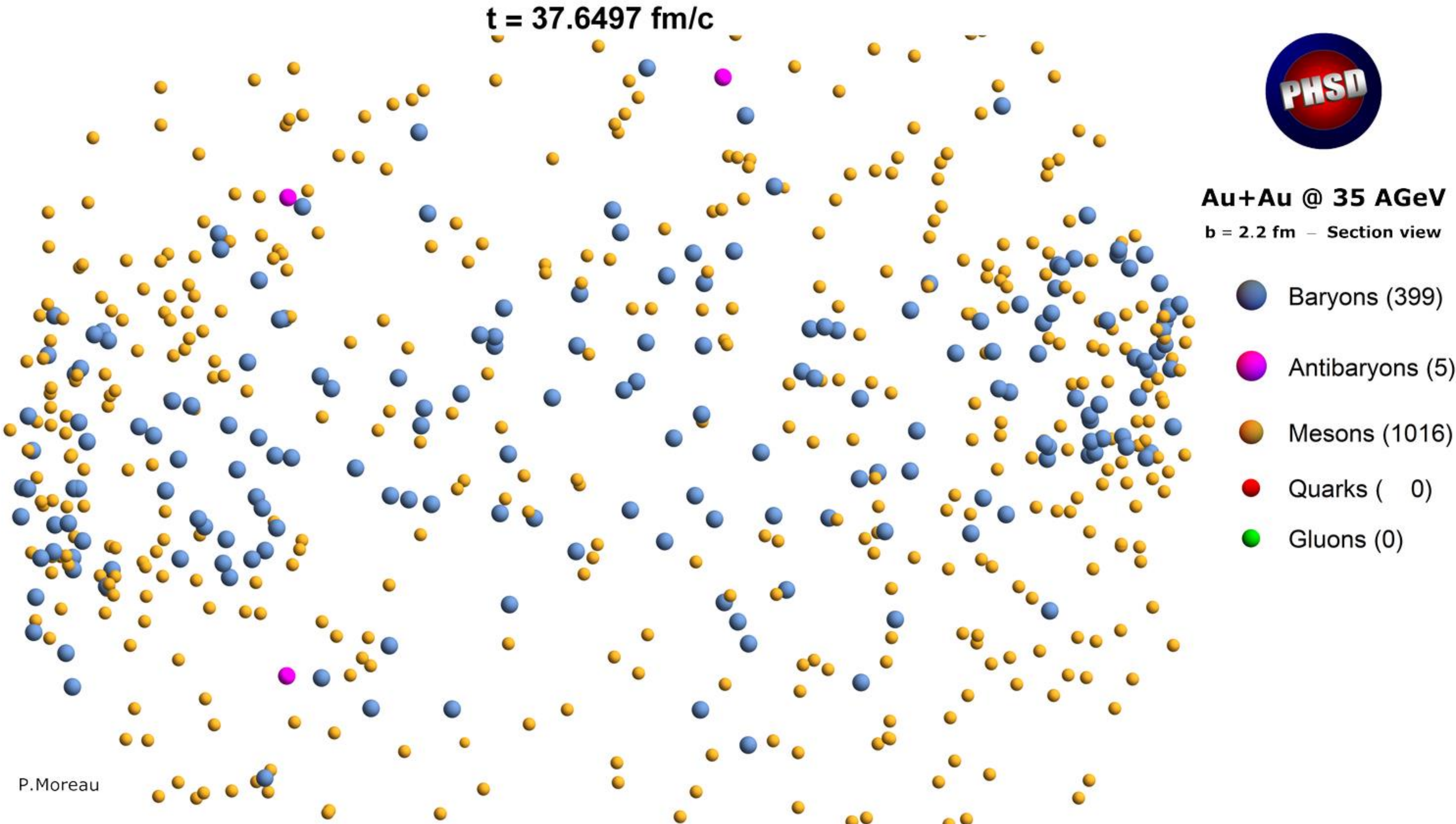

Useful literature

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