Chapter 3: Dynamical Systems

(Ballard: Chapter 5)

Motivation

Natural Processes unfold over time:
- swinging of a pendulum
- decay of radioactive material
- a chemical reaction
- growth of a plant
- formation of a Tornado
- galloping of a horse
- reaching for a cup of tea
- action potential traveling down an axon
- remembering an event

A universal mathematical language for describing processes unfolding in time is dynamical systems theory. An important part of this is the study of differential equations.
Radioactive Decay

Example: Consider a certain amount of radioactive material:

<see applet: http://home.a-city.de/walter.fendt/phe/lawdecay.htm>

The number of atoms that have not decayed yet is a function of time: \( N = N(t) \)

A simple Differential Equation

Let’s treat \( N(t) \) as a continuous function (neglecting the fact that there are discrete atoms decaying). Good approximation if \( N \) is large (typically, you’re dealing with \( 10^{23} \) of them).

Then we can speak of the (temporal) derivative of \( N(t) \), i.e. the slope of the \( N(t) \) graph.

different notations:

\[
N'(t) \equiv \frac{dN(t)}{dt} \equiv \dot{N}(t) \equiv \lim_{\Delta t \to 0} \frac{N(t + \Delta t) - N(t)}{\Delta t}
\]

Physics: at any time this derivative is proportional to number of remaining atoms. Expressing this fact mathematically we write:

\[
\dot{N}(t) = -\alpha N(t) \quad \alpha > 0
\]

This is a simple differential equation (DE), relating a function to its derivative!
Review: derivative of functions

\[ f(t) = 4t^3 + t^2 - 8t - 7 \quad \Rightarrow \quad f'(t) = df(t)/dt = 12t^2 + 2t - 8 \]

\[ g(t) = 4 \sin(t) + 5 \quad \Rightarrow \quad g'(t) = dg(t)/dt = 4 \cos(t) \]

\[ h(t) = \exp(ct), \text{ c is constant} \quad \Rightarrow \quad h'(t) = dh(t)/dt = c \exp(ct) \]

Back to radioactive decay

\[ \dot{N}(t) = -\alpha N(t) \]

In general: \[ \dot{x}(t) = f \left( x(t), t \right) \]

What is the set of functions for which this equation holds? Answering this means to “solve” the differential equation.

Let’s try to guess one or two:

\[ N_1(t) = A - Bt \quad : \quad -B = -\alpha (A - Bt) \quad \text{NO!} \]

\[ N_2(t) = \exp(-\alpha t) \quad : \quad -\alpha \exp(-\alpha t) = -\alpha \exp(-\alpha t) \quad \text{YES!} \]

But is this the only one? No: \( N(t) = A \exp(-\alpha t) \) with arbitrary \( A \) works too!
Back to radioactive decay

N(t) = A \exp(-\alpha t) is general solution to the DE, \( \dot{N}(t) = -\alpha N(t) \)
there are no others (try finding them)

To also specify the correct A we need more information, e.g. how big is N(t) at time t=0. This is called an initial value problem (IVP).

Assume we know that N(0) = 1000.
How do we have to choose A?

\( \text{Let's compute N}(0): \ N(0) = A \exp(-\alpha t) = A \)

Since N(0) is supposed to be 1000 we need to set A=1000.
Thus: the specific solution to IVP is N(t) = 1000 \exp(-\alpha t)

Example: assume \( \alpha = 1 / 500 \text{years} \). What is N(70 years)?
N(70 years) = 1000 \exp(\ - \ (1 / 500 \text{years}) \times 70 \text{years})
= 1000 \exp(-70/500) \approx 869

Example continued: N(t) = A \exp(-\alpha t) , \( \alpha = 1 / 500 \text{years} \)

Question: What is the time after which N(t) is \( \frac{1}{2} N(0) \) ? (half-life \( T_{1/2} \))

Answer:
when \( \exp(-\alpha T_{1/2}) = \frac{1}{2}, \ i.e. \)

\( T_{1/2} = \ln(1/2)/(-\alpha) \approx 347 \text{ years} \)

To play some more with radioactive decay, visit:
http://www.safety.ubc.ca/rad/calc/calcframe.htm
**Graphical Interpretation of DE**

\[ N(t) = -\alpha N(t) \]

_Idea:_ read this as a prescription of how to choose \( dN(t)/dt \) (the slope of \( N(t) \)) as a function of \( N(t) \).

Solving the DE means to find smooth curves that have these line segments as tangents, there’s one curve for any \( A \).

**Second Graphical Interpretation**

\[ \dot{N}(t) = -\alpha N(t) \]

_Idea:_ read this as a prescription of how \( N(t) \) “moves” as a function of \( N(t) \).

At any time:
- If \( N \) is positive (right side) \( N \) will be shrinking at that time (left arrow).
- If \( N \) is negative (left side) \( N \) will be growing at that time (right arrow) because \( dN/dt \) will be positive. (Negative number of atoms does not make a lot of sense but the equation does not care!)

At \( N=0 \), \( dN/dt=0 \), i.e. \( N \) does not change at all: _fixed point_
Another simple example

Consider a rocket being accelerated by a constant force. According to physical laws this situation is described by the following simple differential equation:

\[
dv(t)/dt = -g + F/m
\]

where \(v(t)\) is the velocity of the rocket, \(m\) is its mass, \(g\) is the acceleration due to gravity \((g \approx 9.81 \text{ m/s}^2)\) and \(F\) is the force the engine generates.

To find a solution \(v(t)\) we have to find a function whose derivative is \(-g + F/m\).

**Answer:** the solutions are \(v(t) = (-g+F/m)t + v_0\), where \(v_0\) is an arbitrary initial velocity.

Let’s verify

DE: \(dv(t)/dt = -g + F/m\)

proposed solution: \(v(t) = (-g+F/m)t + v_0\)

**Verify** : compute the derivative of the proposed \(v(t)\) and see if DE holds

\[
dv(t)/dt = -g + F/m \quad \text{O.k., works!}
\]

**Side question:** How big does the force of the rocket have to be to assure lift-off? \(F > mg\)

Recall the \(N(t)\) picture for radioactive decay. What does the picture look like in the rocket case?
Differential Equations

In general: \( x(t) = f(x(t), t) \)

**Problem:** can be very tough or even impossible to solve analytically

**Three Approaches:**

- **numerical simulation:** generate approximate solutions with computer
- **local stability analysis:** study fixed points and surrounding areas
- **global stability analysis:** look for an "energy" function governing global behavior

Next time: all this for vectors

Approximate Numerical Solution

Note: not covered in Ballard’s book!

\[ \dot{x}(t) = f(x(t), t) \]

**Idea:** approximate derivative with finite ratio: \( \dot{x}(t) \approx \frac{x(t + \Delta t) - x(t)}{\Delta t} \)

Thus: \( \frac{x(t + \Delta t) - x(t)}{\Delta t} \approx f(x(t), t) \)

Now solve for \( x(t + \Delta t) \): \( x(t + \Delta t) = x(t) + f(x(t), t)\Delta t \)

Given an initial condition \( x(0) \) we can select a “suitable” \( \Delta t \) and compute \( x(t + n\Delta t) \) in an iterative manner, \( n=1,2,3,... \)

This is the so-called **Euler method.**
Graphical Interpretation of Euler method

\[ x(t + \Delta t) = x(t) + f(x(t), t)\Delta t \]

**Idea:** extrapolate tangents to \( x(t) \) for finite distance

\[ x(t) \]

true \( x(t_0 + \Delta t) \)

error

estimated \( x(t_0 + \Delta t) \)

t

- **Example:** \( \dot{N}(t) = -\alpha N(t), \quad \alpha > 0 \)

**Prescription:** \( x(t + \Delta t) = x(t) + f(x(t), t)\Delta t \)

\[ \Rightarrow N(t + \Delta t) = N(t) - \alpha N(t)\Delta t = N(t)(1 - \alpha \Delta t) \]

**Notes:**

The right choice of \( \Delta t \) is problematic:
- if \( \Delta t \) too small, simulations take too long
- if \( \Delta t \) too big, simulations can be grossly incorrect
- good heuristic is to halve \( \Delta t \) and see if result remains the same

- Many other methods, Euler method is just the simplest
Excursion:
Taylor expansion / Taylor series

Idea: locally approximate an arbitrary function by a polynomial

\[ f(x_0 + \xi) \approx f(x_0) + \frac{1}{1!} f'(x_0)\xi + \frac{1}{2!} f''(x_0)\xi^2 + \cdots \]

Example: approximate around \( x=0 \)

\[ f(x) = \sin(x) \]

\[ \sin(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \cdots \]

Frequently, only the first 2 terms used: linearization,

\[ \text{e.g. } \sin(x) \approx x \text{ is the linearization of } \sin(x) \text{ around } x=0 \]

Question: How good is approximation?

Answer: The better the more terms

But: for very small \( \xi \), linear term dominates because \( \xi \gg \xi^2 \gg \xi^3 \)
Local Stability Analysis

Idea: Finding solution to DE sometimes very difficult. Stability analysis gets some qualitative insight into behavior more easily.

Consider the logistic DE used to describe population dynamics species:

\[ \dot{x} = (\alpha - 1)x - \alpha x^2, \alpha > 1 \]

\((\alpha-1)x\): growth through reproduction; \(-\alpha x^2\): death due to overcrowding

**Step 1:** find stationary points, points where \(dx/dt=0\)

In the example:

\[ 0 = (\alpha - 1)x - \alpha x^2 \]

has the 2 solutions: \(x_1 = 0; x_2 = (\alpha - 1)/\alpha\)

Kinds of Stationary Points

Stationary point: \(dx/dt=0\)

What happens if system in stationary state gets a small "nudge"

- system returns to stationary point: point is stable (asymptotically stable)
- system runs away from stationary point: point is unstable
- system rests at neighboring point: marginally stable
- saddle point: direction of small nudge matters
logistic DE: \( \dot{x} = (\alpha - 1)x - \alpha x^2, \alpha > 1 \)

**Step 2:** linearize the DE around the stationary points and analyze the stability of the stationary points

**First point:** \( x_1 = 0 \)

\[ x = x_1 + \xi \Rightarrow \dot{x} = \dot{\xi} \]

\( \xi \) is small deviation from stationary point \( x_1 \)

\[ \dot{\xi} = (\alpha - 1)\xi - \alpha \xi^2 \]

**linearization**

\[ \dot{\xi} = (\alpha - 1)\xi \]

This is linear DE for small deviation \( \xi \), telling us whether small deviation is going to grow or decay:

- growth: *unstable fixed point*
- decay: *stable fixed point*

The general solution to the linearized DE is

\[ \xi(t) = \xi(0) \exp((\alpha - 1)t) \]

\[ \lim_{t \to \infty} \xi(t) = \pm \infty \]

In this case we find growth, since \((\alpha - 1) > 0\), i.e. \( x_1 \) is unstable

**Second point:** \( x_2 = \frac{\alpha - 1}{\alpha} \)

\[ x = x_2 + \xi \Rightarrow \dot{x} = \dot{\xi} \]

Rewriting the DE in \( \xi \):

\[ \dot{\xi} = (\alpha - 1)\left(\xi + \frac{\alpha - 1}{\alpha}\right) - \alpha\left(\xi + \frac{\alpha - 1}{\alpha}\right)^2 \]

Simplifying using Taylor expansion:

\[ \dot{\xi} \approx (1 - \alpha)\xi \]

This corresponds to exponential decay since \((1 - \alpha) < 0\), i.e. \( x_2 \) is a stable fixed point