Pattern Formation in Neural Fields

Goal:
Understand how non-linear recurrent dynamics can lead to pattern formation in cortical layers. Discuss relation to visual hallucinations.

Assumptions:
- just a single sheet of neurons
- Stereotypic pattern of lateral excitation (short) + inhibition (longer but not global)
- Wilson-Cowan dynamics
- Reminder: Dale’s law: inhibition mediated by inhibitory interneurons
Basic contribution to field potential at site $x$ from neighbors is given by:

$$e(\tilde{x}, t) \propto \int g(\tilde{x}') S(h(\tilde{x} - \tilde{x}')) d\tilde{x}' \equiv g(\tilde{x}) * S(h(\tilde{x}))$$

where ‘$*$’ is the convolution operator. I.e. to collect all contributions to location $x$ coming from the whole layer we have to integrate over the entire layer.

Recall:

$$f(\tilde{x}) * g(\tilde{x}) \equiv \int f(\tilde{x} - \tilde{x}') g(\tilde{x}') d\tilde{x}'$$

$$f(\tilde{x}) * g(\tilde{x}) = g(\tilde{x}) * f(\tilde{x})$$

$$f(\tilde{x}) * \beta = \beta \int f(\tilde{x}') d\tilde{x}' \quad \text{, for } \beta \text{ constant}$$
The Basic Model Equation

Integro-differential equation (IDE):

\[ \dot{h}(\vec{x}, t) = -h(\vec{x}, t) + p + \gamma g * S(h(\vec{x}, t)) \]

Difference of Gaussian (DoG) kernel:

\[ g(\vec{x}) = \frac{1}{\pi} \left( \exp(-\vec{x}^2) - \frac{1}{2} \exp(-\vec{x}^2/2) \right) \]

Sigmoid non-linearity:

\[ S(h) = \frac{1}{1 + \exp(-h)} \]
\[
\dot{h}(\vec{x}, t) = -h(\vec{x}, t) + p + \gamma g * S(h(\vec{x}, t))
\]

**Step 1**: homogenous stationary solution: \( h_0 = p \)

**Step 2**: linear stability analysis:
- After linearizing and going into Fourier space, equation is linear in Fourier coefficients:

\[
\frac{d}{dt} F[\vec{\epsilon}](k, t) = (2\pi\gamma\omega_p F[g](k) - 1)F[\vec{\epsilon}](k, t)
\]

\[
= \lambda(k)F[\vec{\epsilon}](k, t)
\]

where: \( \omega_p = 1/(e^p(1+e^{-p})^2) \)

Fluctuations can grow if: \( \lambda(k) > 0 \)

The shape of \( \lambda(k) \) follows that of \( F[g](k) \):

\[
F[g](k) = \frac{1}{2\pi} \left( \exp\left(\frac{-k^2}{4}\right) - \exp\left(\frac{-k^2}{2}\right) \right)
\]
\[ \dot{h}(\bar{x}, t) = -h(\bar{x}, t) + p + \gamma g \ast S(h(\bar{x}, t)) \]

The maximum is at $4\ln(2)$, where $F[g](k) = 1/(8\pi)$.

\[ F[g](\vec{k}) = \frac{1}{2\pi} \left( \exp\left(\frac{-\vec{k}^2}{4}\right) - \exp\left(\frac{-\vec{k}^2}{2}\right) \right) \]
\[ \dot{h}(\vec{x}, t) = -h(\vec{x}, t) + p + \gamma \ g \ast S(h(\vec{x}, t)) \]

**Question:**
When is the maximum lambda greater than zero? (That’s when fluctuations will grow.)

**Answer:**
\[ \gamma \geq 4e^p (1 + e^{-p})^2 \]

**Result of our analysis:** instabilities (deviations from homogenous stationary solution) can only grow when \( \gamma \) above curve.
Simulation of system...

\[ h(x, t=0) \]

(initialized per mouse)

\[ h(x, t>0) \]
... leads to:

\[ h(x, t\gg0) \]

\[ S(h(x, t\gg0)) \]
Variety of different patterns:

for different $p, \gamma$ settings: regular and irregular stripes, hexagonal blobs and anti-blobs:

Notes:
- Interesting to note that the exact same dynamics give rise to very different patterns if the parameters are slightly changed.
- The same abstract mechanism of short-range excitation and long-range inhibition is thought to underlie other biological patterns (zebra stripes, leopard spots)
→ reaction diffusion systems
Visual Hallucinations

A variety of drugs induce hallucinations of stereotypical geometric patterns: concentric circles, radial spokes, spirals, checkerboards with expanding check sizes towards periphery.

Ermentrout and Cowan (1979): such patterns can result from blob and stripe patterns on cortical surface, because of non-uniform mapping from retina to cortex.

Kolb et al.  
Eric Schwartz
Mapping from retina to cortex

Retina: polar coordinates $(R, \Theta)$
Cortex: $(\ln(1+R), \Theta)$

- concentric retinal circles A-D mapped onto vertical lines in cortex
- radial spokes E-G in retina mapped onto horizontal lines
- spirals in retina mapped onto oblique lines
Pattern Formation Summary

- neurons wired up in this simple fashion show complex patterns of activity, nice example of \textit{self-organization}

- formation of interesting patterns based on similar mechanisms as in other domains: long-ranging inhibition and short-ranging excitation

- exhibits \textit{spontaneous symmetry breaking}: at macroscopic scale, system initially symmetric. Microscopic deviations from symmetry drive system to final state that is asymmetric at macroscopic scale

- relation to visual hallucinations experienced under certain drugs

- still no plasticity/learning