FIAS Kolloquium

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Fractional charges in solids

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**Fractional charges in one dimension**

trans-polyacetylene \((\text{CH})_n\):

2-fold degenerate ground state:

Peierls distortion

kink or soliton: from A \(\rightarrow\) B

\(1\ \pi\)-electron per C atom

\[ \begin{align*}
\text{C} & \quad \text{C} & \quad \text{C} & \quad \text{C} \\
\text{C} & \quad \text{C} & \quad \text{C} & \quad \text{C}
\end{align*} \]

undistorted \(\rightarrow\) distorted + kink

\(\omega\)

\(N(\omega)\)

Su + Schrieffer

contribution to kink: 50 % occ. + 50 % unocc.

kink state empty: charge e is missing; spin zero

kink state singly occ.: neutral spin \(\frac{1}{2}\)

spin-charge separation!
Spinless fermions

- **kink**: ½ state from occupied band
- ½ state from empty band
- **kink state empty**: charge $-e/2$ is missing
- **occupied**: charge $e/2$ is added
- **fractional charge**: $\pm e/2$
**Two dimensions: fractional quantum Hall effect**

\( R_H \) (or \( \rho_{xy} \)) has pronounced steps at filling factors \( \nu \) of Landau levels

\[
\phi = B \cdot F = N_{\ell} \cdot \phi_0
\]

flux quantum \( \phi_0 = \frac{hc}{e} \)

\[
\nu = \frac{N_{e\ell}}{N_{f\ell}}
\]

- \( \nu = 1, 2, 3, \ldots \) \( \rightarrow \) integer QHE
- \( \nu = 1/3, 1/5 \ldots \) \( \rightarrow \) FQHE
FQHE: due to electron-electron interactions

\[ \nu = 1/3 \quad \rightarrow \quad 3 \times \text{as many flux quanta} \quad \phi_0 = \frac{hc}{e} \quad \text{as there are electrons} \]

idea: mapping of the interacting electron system onto a system of charged bosons \( \rightarrow \) electrons with attached \( \frac{1}{\nu} = 2k + 1 \) flux tubes

Why bosons? - because of symmetry properties

consider \( \nu = 1/3 \quad \text{curl} \ A(x) = 3\phi_0 \rho(x) \quad \text{flux is tied to particle at} \quad x = 0 \)

exchange of two flux tubes \( \rightarrow \) (-1) bosons

electrons are well separated because of uniform field distribution
bosons do not experience mean field but only \textit{gauge interactions} with other bosons

\begin{itemize}
  \item formations of Bose-Einstein condensate  \textit{charged} superfluid
  \item incompressible electron system since $\delta \rho(x)$ implies $\delta A(x)$
  \item \textit{Meißner Effekt} prevents $\delta A(x)$ from penetrating
\end{itemize}

increase of magnetic field:

\begin{itemize}
  \item addit. flux quantum  \textit{charge deficit} 1/3 electron  \textit{hole}
  \item one flux quantum less  \textit{charge surplus} 1/3 electron
\end{itemize}

\begin{itemize}
  \item excitations with charge $\pm e/3$
\end{itemize}

\textbf{statistics:} each flux quantum $\phi_0$ carries associated charge $\pm ve$

exchange of a pair yields factor $e^{i\pi\nu}$ \textit{fractional statistics}
Here 3D: **pyrochlore lattice** at half-filling, fully spin polarized electrons (spinless fermions)

\[ H = -t \sum_{i,j} (c_i^+ c_j + \text{h.c.}) + V \sum_{i,j} n_i n_j \]

To show: excitations with fractional charges \( \pm \frac{e}{2} \)

**Pyrochlore checkerboard**

\( V \gg t \)

**Tetrahedron rule**

2 empty and 2 occupied sites
Checkerboard lattice: 

ground-state degeneracy: 

\[ N_{\text{deg}} = \left( \frac{4}{3} \right)^{3N} \]

\( N = \text{number of sites} \)

t = 0

two configurations:

loops

related to ice model

(Pauling)

solid lines connect occupied sites

a theory of strings!
Addition of an electron → decay into two excitations with charge e/2 (backflow)

pyrochlore lattice:
Vacuum fluctuations

generation of pairs with fractional charges \( +e/2, \ -e/2 \)

excitation energy: \( \Delta E = V \)
finite hopping \( t \neq 0 \):

2nd order: constant energy contribution

3rd order: effective Hamiltonian \( g = \frac{12t^3}{V^2} \) (sign is irrelevant)

\[
H_{\text{eff}} = g \sum_{\{\Theta\}} \left( \left| \begin{array}{c} \Theta \\ \Theta \end{array} \right> \left< \Theta \right| - \left| \Theta \right> \left< \Theta \right| + \text{H.c.} \right)
\]

relative sign problem:

lifting of the macroscopic degeneracy
Confinement of fractional charges

change in kinetic energy in the presence of two fixed charges $e/2$

site $i$:  

$$\varepsilon_i = -\frac{1}{6} \sum_{\Omega/i\varepsilon} \langle \bar{\psi}_0 (0, \mathbf{r}) | H_{\text{eff}} | \psi_0 (0, \mathbf{r}) \rangle$$

constant confining force

reason: vacuum fluctuations are reduced in the vicinity of the string

numerics:  

$$\Delta \varepsilon_{\text{kin}} \simeq 0.2 \, g \cdot r$$  

$r = \text{units of a}$

$$r > r_c \simeq 0.4 \left( \frac{V}{t} \right)^3$$

since  

$$\Delta \varepsilon_{\text{kin}} > V$$

pair production  

$$\left( \frac{e}{2}, -\frac{e}{2} \right)$$
further comments:

- since \[ g = \frac{12t^3}{V^2} \ll t \] huge (extended) quasiparticles
  e.g., \[ d \sim 100a \]

- with increasing doping transition to \( e/2 \) plasma
vacuum polarization:

\[ \delta n_i = \langle \bar{\psi}_0 (0, r) | n_i | \bar{\psi}_0 (0, r) \rangle - \langle \psi_0 | n_i | \psi_0 \rangle \]

red = increase of charge

blue = decrease

energy increase due to polarization of vacuum
Lattice gauge theory

dual lattice: checkerboard → square
pyrochlore → diamond

particles sit on links!

checkerboard lattice:

½ filling → loop covering
mapping of the model onto a U(1) lattice gauge theory

extension of Hilbert space to include links with \( n_j(x) \) particles and conjugated phase \( \phi_j(x) \in [0, 2\pi] \)

elimination of unphysical states in Hamiltonian:

\[
H_{\text{phys}} = \lim_{w \to \infty} W \sum_x \sum_{j=1}^{2} \left( \left( \hat{n}_j(x) - \frac{1}{2} \right)^2 - \frac{1}{4} \right)
\]

phase operator generates ladder operators

\[
e^{\pm i\phi_j(x)} |n_j\rangle = |n_j \pm 1\rangle
\]
\[ H_{\text{eff}} = g \sum_{\{ \text{oriented links} \}} \left( | \begin{array}{c} \vdots \end{array} \rangle \langle \begin{array}{c} \vdots \end{array} | - | \begin{array}{c} \vdots \end{array} \rangle \langle \begin{array}{c} \vdots \end{array} | \right) + \text{H.c.} \]

goes over into

\[ H_{\text{eff}} = \lim_{W \to \infty} W \sum_{x,j} \left( \hat{n}_j(x) - \frac{1}{2} \right)^2 - 2g \sum \cos \left( \sum \pm \phi \right) \]

for bipartite lattice: oriented links

\[ \hat{E}_j(x) = (-1)^{x_1+x_2} \left( \hat{n}_j - 1/2 \right) \quad \quad \hat{A}_j(x) = (-1)^{x_1+x_2} \phi_j \]

densely packed loops (tetrahedron rule):

Gauß’s law \[ [ \Delta \hat{E}_j(x) - \rho(x) ]|_{\text{phys}} = 0 \] with \( \rho(x) = 0 \) since each point is touched by two dimers
together with Gauß's law we obtain (up to a constant)

\[ H_{\text{eff}} = \lim_{w \to \infty} W \sum_{x,j} \hat{E}_j^2(x) - 2g \sum_x \cos \left( \sum_{\mathcal{O}} \hat{A}_j(x) \right) \]

gauge field \( \hat{A}_j(x) = \pm \hat{\phi}_j(x) \), \( \mathcal{O} \) contains point \( x \)

like Hamiltonian of compact (2+1) dim. QED

(Polyakov)

confinement

but here: half-integer \( E \) fields

ground state is accessible by Monte Carlo calculation (path intergrals)
Perspectives:

- Inclusion of spin:
  \[ H_w = J \sum_{\langle ij \rangle} S_i S_j \]
  spin is highly nonlocal
  carried by "gluon"

- Statistics of e/2 charges

- 3D pyrochlore lattice: U(1) gauge theory allows for deconfined charges