Superfluid turbulence
in an Ultracold Bose Gas

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Equilibration

Transient, metastable state
e.g. Turbulence
Non-thermal fixed point
Classical Turbulence

Kinetic energy cascade

large scales (source)
→ small scales (sink)
“Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity.”

(Richardson, 1920)
Classical Turbulence

Kinetic energy cascade
large scales (source) → small scales (sink)

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and little whirls have lesser whirls and so on to viscosity.”
(Richardson, 1920)

Kolmogorov (1941)  \[ E(k) \sim k^{-5/3} \] (dynamical critical phenomenon)
Experiments

\[ \phi_1 = 2E_{||} \]

\[ \frac{k}{k_d} = k\eta \]
Wave turbulence
Wave Turbulence

Example: Driven surface waves on water

[H. Xia et al., EPL 91 (10) 14002]
Transport in momentum space

Imagine you had a balance equation for the radial flux

\[ \partial_t n(k) = - \partial_k Q(k) \]
Transport in momentum space

Transport equation (Quantum Boltzmann eq.):

\[
\frac{\partial}{\partial t} n(k) = - \partial_k Q(k) \sim k^{d-1} J(k)
\]

\[
= k^{d-1} d\Omega_k \int \frac{d^d p \, d^d q \, d^d r \, |T_{k p q r}|^2 \delta(k + p - q - r) \delta(\omega_k + \omega_p - \omega_q - \omega_r)}{
\text{coupling} \quad \text{mom. conservation} \quad \text{energy conservation}}
\]

\[\times \left[(n_k + 1)(n_p + 1)n_q n_r - n_k n_p (n_q + 1)(n_r + 1)\right]\]

\text{in-scattering rate} \quad \text{out-scattering rate}
Transport in momentum space

Radial transport equation (Quantum Boltzmann):

\[ \partial_t n(k) = - \partial_k Q(k) \sim k^{d-1} J(k) \]

\[ = k^{d-1} d\Omega_k \int \frac{d^dp \, d^dq \, d^dr}{|T_{kpqr}|^2} \delta(k + p - q - r) \delta(\omega_k + \omega_p - \omega_q - \omega_r) \]

\[ \times [(n_k + 1)(n_p + 1)n_q n_r - n_k n_p (n_q + 1)(n_r + 1)] \]

Stationary distribution \( n(k, t) \equiv n(k) \) if \( Q(k) \equiv Q \)

This requires a particular scaling of \( n(k) \sim k^{-\zeta} \)
Wave turbulence

Stationary scaling $n(k)$ within **inertial** region:

$$\log n(k) \sim k^{-\zeta}$$
Wave Turbulence

Example: Driven surface waves on water

Theory prediction:

\[ E_\omega \sim \omega^{-17/6}. \]

[Zakharov & Filonenko (67)]

[H. Xia et al., EPL 91 (10) 14002]
Wave turbulence in an ultracold Bose gas
ULTRAcool...

... atoms @ nanokelvins -

trapped only a few mm away from

glass cell @ room temperature

(vacuum of $10^{-12}$ Torr, i.e. $10^{-15}$ bar, or $10^{-10}$ Pa, ≈ atmospheric pressure on the moon)
Superfluid dilute ultracold Bose Gas

Gross-Pitaevskii Equation:

\[ i \frac{\partial \Psi(\rho, t)}{\partial t} = \left( -\frac{\nabla^2}{2} + g|\Psi(\rho, t)|^2 \right) \Psi(\rho, t) \]

Momentum spectrum:

\[ n(k) = \langle \Psi^*(k) \Psi(k) \rangle \]
Bose gas in $d$ spatial dimensions

\[ n \sim k^{-\zeta} \]

\[ n \]

\[ \zeta = d \]

\[ \text{momentum } k \]

C. Scheppach, J. Berges, TG  PRA 81 (10) 033611
Bose gas in $d$ spatial dimensions $n \sim k^{-\zeta}$

\[ n \sim k^{-\zeta} \]

\[ \zeta = d \]

(thermal equilibrium)

\[ \zeta = 2 \]

(Rayleigh-Jeans)

C. Scheppach, J. Berges, TG PRA 81 (10) 033611
Transport in momentum space

Quantum Boltzmann breaks down for large $n$, once $T^2 n^3 \gg O(1)$

$$\partial_t n(k) = -\partial_k Q(k) \sim k^{d-1} J(k)$$

$$= k^{d-1} d\Omega_k \int d^dp \, d^dq \, d^dr \, |T_{kpqr}|^2 \delta(k + p - q - r) \delta(\omega_k + \omega_p - \omega_q - \omega_r)$$

coupling  mom. conservation  energy conservation

$$\times [(n_k + 1)(n_p + 1)n_q n_r - n_k n_p (n_q + 1)(n_r + 1)]$$

in-scattering rate  out-scattering rate
Bose gas in $d$ spatial dimensions

$n \sim k^{-\zeta}$

New exponent beyond Quantum Boltzmann!

$n \sim k^{-\zeta}$

$\zeta = d + 2$

$\zeta = d$

momentum $k$

$n$

J. Berges, A. Rothkopf, J. Schmidt, PRL 101 (08) 041603
C. Scheppach, J. Berges, TG PRA 81 (10) 033611
Vortices in a superfluid ultracold Bose gas
Superfluid hydro of Bose-condensed Gas

The Gross-Pitaevskii Eq. in the classical regime,

\[ i \frac{\partial \Psi(\rho, t)}{\partial t} = \left( -\frac{\nabla^2}{2} + g|\Psi(\rho, t)|^2 \right) \Psi(\rho, t) \]

is equiv. to

\[ \frac{\partial}{\partial t} n + \nabla \cdot (n\mathbf{u}) = 0 \]
\[ \frac{\partial}{\partial t} \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla Q \]

(Euler eq.)

with defs.

\[ \Psi(\rho, t) = \sqrt{n(\rho, t)} \exp[i \varphi(\rho, t)] \]
\[ Q = gn \]
\[ \mathbf{u}(\rho, t) = \nabla \varphi(\rho, t) \]
Quantum Vortex: Discrete Velocities

\[ \Psi(\rho, t) = \sqrt{n(\rho, t)} \exp[i \varphi(\rho, t)] \]

*complex field*

\[ u(\rho, t) = \nabla \varphi(\rho, t) \]

*velocity*
\[ \Psi(\rho, t) = \sqrt{n(\rho, t)} \exp[i\varphi(\rho, t)] \]

\[ u(\rho, t) = \nabla \varphi(\rho, t) \]

\[ \text{[c] Möttönen, Helsinki; PRL 99 (07) } \]
\[ \Psi(\rho, t) = \sqrt{n(\rho, t)} \exp[i \varphi(\rho, t)] \quad u(\rho, t) = \nabla \varphi(\rho, t) \]

Density \( n \) and phase \( \varphi \)

[W. P. Reinhardt, Seattle]

[Wolfgang Ketterle, MIT]
Vortices in a Na condensate

Fig. 4. Formation and decay of a vortex lattice. The condensate was rotated for 400 ms and then equilibrated in the stationary magnetic trap for various hold times. (A) 25 ms, (B) 100 ms, (C) 200 ms, (D) 500 ms, (E) 1 s, (F) 5 s, (G) 10 s.

J. R. Abo-Shaeer, C. Raman, J. M. Vogels, W. Ketterle
20 APRIL 2001  VOL 292  SCIENCE
Vortex pairs

[T.W. Neely et al. PRL 104 (10)]
Helmholtz Vortex Law

[T.W. Neely et al. PRL 104 (10)]
Vortex tangles in Bose Einstein Condensates

[E.A.L. Henn et al. PRL 103 (09)]

[N. Berloff & B. Svistunov, PRA (02)]
Movie 1: Phase evolution

\[ \Psi(\rho, t) = \sqrt{n(\rho, t)} \exp[i \varphi(\rho, t)] \]
Movie 2: Vortex “gas” & Spectrum

\[ n(k) = \langle \Psi^*(k) \Psi(k) \rangle \big|_{\text{angle average}} \]
Spectrum in 2+1 D

Occupation number $n(k)$

Radial momentum $k$
Spectrum in 2+1 D

Radial momentum $k$
Spectrum in 2+1 D
Spectrum in 2+1 D

Occupation number $n(k)$ vs Radial momentum $k$

Lattice site $x$

Lattice site $y$

$n(k)$ vs $k^{-4}$

$n(k)$ vs $k^{-2}$
Non-thermal fixed point

\[ n(t,p) \sim p^{-\alpha} \]

initial conditions
\[ n(t=0,p) \]

thermal equilibrium
\[ n_{BE}(p) \]

[Fig.: Berges 08]
Solitons in 1 spatial dimension
Relativistic scalar field
Strong Turbulence

Simulations of the non-linear Klein-Gordon equation, \( O(2) \) symmetry

\[
(\partial^2_t - \partial^2_x) \varphi(x, t) + \lambda \varphi^3(x, t) = 0
\]

Initial condition: Highly occupied zero mode, Unoccupied modes with \( k>0 \)

(video)

See also: http://www.thphys.uni-heidelberg.de/~sefty/videos

Strong Turbulence = Charge Separation

Modulus of complex field $|\varphi|$ vs. mean charge distribution

cf. also Tkachev, Kofman, Starobinsky, Linde (1998)
Strong Turbulence = Charge Separation

Charge density distribution vs. power spectrum

\((d = 2, N = 2)\)

Visualization of spiral and scroll waves in simulated and experimental cardiac tissue

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Far field pacing supersedes anti-tachycardia pacing in a generic model of excitable media

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\[
\frac{\partial u}{\partial t} = \varepsilon^{-1} u(1-u) \left( u - \frac{v+b}{a} \right) + \nabla^2 u
\]

Barkley model
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Lewis Fry Richardson, FRS (1881-1953)

Big whirls have little whirls that feed on their velocity,
and little whirls have lesser whirls and so on to viscosity.

(L.F. Richardson, *The supply of energy from and to Atmospheric Eddies*, 1920)

Great fleas have little fleas upon their backs to bite 'em,
And little fleas have lesser fleas, and so ad infinitum.
And the great fleas themselves, in turn, have greater fleas to go on;
While these again have greater still, and greater still, and so on.

(Augustus de Morgan, *A Budget of Paradoxes*, 1872, p. 370)

So, naturalists observe, a flea
Has smaller fleas that on him prey;
And these have smaller still to bite 'em;
And so proceed ad infinitum.

(Jonathan Swift: *Poetry, a Rhapsody*, 1733)