Flow velocities in dissipative relativistic hydrodynamics – Eckart and/or Landau-Lifshitz –

do we have a choice?

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– Flow-frame dependencies
– Dissipative hydro with general flow-frames
– Generic stability: a benchmark

with Tamás Biró

Dissipative hydrodynamics

$$\partial_\mu N^\mu = 0, \qquad \partial_\nu T^{\mu\nu} = 0$$ 5

$$N^\mu = nu^\mu + j^\mu$$ 4

$$T^{\mu\nu} = eu^\nu u^\mu + q^\nu u^\mu + q^\mu u^\nu - p\Delta^{\mu\nu} + \Pi^{\mu\nu}$$ 10 +3 (u^μ)

$$u^\mu u_\mu = 1,\ j^\mu u_\mu = 0,\ q^\mu u_\mu = 0,\ \Pi^{\mu\nu} u_\mu = 0$$

$$j^\mu, q^\nu, \Pi^{\mu\nu}\ ?$$ 3+3+6=12

Interdependent theoretical problems:

- causality,
- generic stability,
- flow-frames (Eckart, Landau-Lifshitz, etc.),
- dissipation.

Flow-frames (u^μ) and dissipation

Perfect:

$$N^\mu = nu^\mu$$
$$T^{\mu\nu} = eu^\nu u^\mu - p\Delta^{\mu\nu}$$

Dissipative:

$$N^\mu = nu^\mu + j^\mu$$
$$T^{\mu\nu} = eu^\nu u^\mu + q^\nu u^\mu + q^\mu u^\nu - p\Delta^{\mu\nu} + \Pi^{\mu\nu}$$

Landau-Lifshitz:

$$N^\mu = \hat{n}\hat{u}^\mu + \hat{j}^\mu$$
$$T^{\mu\nu} = \hat{e}\hat{u}^\nu \hat{u}^\mu - \hat{p}\hat{\Delta}^{\mu\nu} + \hat{\Pi}^{\mu\nu}$$

Eckart:

$$N^\mu = nu^\mu$$
$$T^{\mu\nu} = eu^\nu u^\mu + q^\nu u^\mu + q^\mu u^\nu - p\Delta^{\mu\nu} + \Pi^{\mu\nu}$$

partial perfection….

$$u^\mu = \frac{\hat{u}^\mu + \hat{z}^\mu}{\zeta}$$

$$\hat{u}^\mu \hat{z}_\mu = 0, \quad \frac{1}{\zeta} = \hat{u}^\mu u_\mu = \frac{1}{1-\hat{z}^2},$$

u^μ Eckart velocity

$\hat{u}^\mu$ Landau-Lifshitz velicity

$\hat{z}^\mu$ E-LL relative four velocity

$$N^\mu = \hat{n}\hat{u}^\mu + \hat{j}^\mu = nu^\mu \quad \Rightarrow \quad \hat{n} = \frac{n}{\zeta}, \quad \hat{j}^\mu = nu^\mu - \hat{n}\hat{u}^\mu = n\frac{\hat{u}^\mu + \hat{z}^\mu}{\zeta} - \frac{n}{\zeta}\hat{u}^\mu = \hat{n}\hat{z}^\mu$$

Landau-Lifshitz:

$$N^{\mu} = \hat{n}\hat{u}^{\mu} + \hat{j}^{\mu}$$

$$T^{\mu\nu} = \hat{e}\hat{u}^{\nu}\hat{u}^{\mu} + \hat{P}^{\mu\nu} = \hat{e}\hat{u}^{\nu}\hat{u}^{\mu} - \hat{p}\hat{\Delta}^{\mu\nu} + \hat{\Pi}^{\mu\nu}$$

Eckart:

$$u^{\mu} = \frac{\hat{u}^{\mu} + \hat{z}^{\mu}}{\zeta}$$

$$N^{\mu} = n u^{\mu}$$

$$T^{\mu\nu} = e u^{\nu} u^{\mu} + q^{\nu} u^{\mu} + q^{\mu} u^{\nu} - p\Delta^{\mu\nu} + \Pi^{\mu\nu}$$

$$e = \frac{1}{\zeta^2}\left(\hat{e} + \hat{p}\hat{z}^2 + \hat{z}_{\nu}\hat{\Pi}^{\nu\mu}\hat{z}_{\mu}\right)$$

$$q^{\mu} = \frac{1}{\zeta}\left(\hat{e}\hat{u}^{\mu} - e(\hat{u}^{\mu} + \hat{z}^{\mu}) + \hat{P}^{\mu\nu}\hat{z}_{\nu}\right)$$

$j^{\mu}, q^{\nu}, \Pi^{\mu\nu}$

dissipative ?

Inviscid Landau-Lifshitz fluid: $\hat{P}^{\mu\nu} = -\hat{p}\hat{\Delta}^{\mu\nu}, \hat{j}^{\mu} \neq 0^{\mu}$

$$e = \frac{1}{\zeta^2}\left(\hat{e} + \hat{p}\hat{z}^2\right) \Rightarrow e + \hat{p} = h = \frac{\hat{h}}{\zeta^2}$$

$$q^{\mu} = \frac{\hat{e} + \hat{p}}{\zeta^3}\left((\zeta^2 - 1)\hat{u}^{\mu} - \hat{z}^{\mu}\right) \Rightarrow \frac{q^{\mu}}{h} = \left(\zeta - \frac{1}{\zeta}\right)\hat{u}^{\mu} - \frac{\hat{j}^{\mu}}{n}$$

$$\hat{j}^{\mu} = \hat{n}\hat{z}^{\mu}$$

Perfect Landau-Lifshitz(-Eckart):

$$N_0^{\ \mu} = nu^\mu$$
$$T_0^{\ \mu\nu} = eu^\nu u^\mu - p\Delta^{\mu\nu}$$

$$u^\mu = \frac{\hat{u}^\mu + \hat{z}^\mu}{\zeta}$$

$$N_0^\mu = \hat{n}\hat{u}^\mu + j^\mu$$
$$T_0^{\ \mu\nu} = \hat{e}\hat{u}^\nu\hat{u}^\mu + q^\nu\hat{u}^\mu + q^\mu\hat{u}^\nu - p\hat{\Delta}^{\mu\nu} + \Pi^{\mu\nu}$$

$$\hat{n} = \frac{n}{\zeta}, \quad j^\mu = \frac{n\hat{z}^\mu}{\zeta}$$

$$\hat{e} = \frac{e+p}{\zeta^2} - p, \quad q^\mu = (e+p)\hat{z}^\mu, \quad \Pi^{\mu\nu} = \frac{\hat{z}^\mu\hat{z}^\nu}{e+p}$$

Frame dependent parts are frame dependent.

Perfect fluid is a class of $N^\mu, T^{\mu\nu}$.

Flow-frames and entropies

Ideal:

$$S_0^\mu(T_0^{\mu\nu}, N_0^\mu) = s_0(e_0, n_0)u^\mu = \left(\beta_0(e_0 + p_0) - \alpha_0 n_0\right)u^\mu = \beta_\nu T_0^{\mu\nu} - \alpha N_0^\mu + p_0\beta^\mu$$

$$\beta^\mu = \frac{u^\mu}{T}$$

First order (Eckart):

$$S^\mu(T^{\mu\nu}, N^\mu) = s(e,n)u^\mu + \frac{q^\mu - \mu j^\mu}{T} = \left(\beta(e+p) - \alpha n\right)u^\mu + \beta q^\mu - \alpha j^\mu =$$

$$= \beta\left((e+p)u^\mu + q^\mu\right) - \alpha(nu^\mu + j^\mu) = \beta_\nu T^{\mu\nu} - \alpha N^\mu + p\beta^\mu$$

Second order (Israel–Stewart):

$$j^\mu$$

$$S^\mu(T^{\mu\nu}, N^\mu) = \left(s(e,\, n) - \frac{\beta_0}{2T}\Pi^2 - \frac{\beta_1}{2T}q_\nu q^\nu - \frac{\beta_2}{2T}\pi^{\nu\sigma}\pi_{\nu\sigma} \right)u^\mu + \frac{1}{T}\left(q^\mu + \alpha_0 \Pi q^\mu + \alpha_1 \pi^{\mu\nu} q_\nu\right)$$

Flow-frames and dissipation in Eckart theory (general frame)

$$Tds + \mu dn = de$$

$$\partial_a S^a = \dot{s}(e,n) + s\partial_a u^a + \partial_a J^a \geq 0$$

$$J^\mu = \beta q^\mu - \alpha j^\mu$$

$$u^\mu \partial_\nu T^{\mu\nu} = \dot{e} + e\partial_\mu u^\mu + \partial_\mu q^\mu + u_\mu \dot{q}^\mu + u_\mu \partial_\nu P^{\mu\nu} = 0$$

$$\partial_\mu N^\mu = \dot{n} + n\partial_\mu u^\mu + \partial_\mu j^\mu$$

$$\sigma = -j^\mu \partial_\mu \alpha + \beta \Pi^{\mu\nu} \partial_\nu u_\mu + q^\mu \left(\partial_\mu \beta - \beta \dot{u}_\mu\right) \geq 0$$

Eckart term

Origin of the concept of perfection

Dissipation and transport: frame dependent.

Summary of flow-frames (u^μ) dependencies:

Dissipative and non-dissipative parts
of energy-momentum and
conserved charge densities.
Entropy four vectors.
Entropy production (dissipation).
Perfect fluid.

} flow-frame dependent

+ e.g. gradient expansion (of Baier-Romatschke)
+ stability…

Thermodynamics requires flow-frames?

Kinetic theory?

$$k^{\mu}\partial_{\mu}f = C(f)$$

Boltzmann equation

$$k^{\mu}k_{\mu} = m^2$$

Boltzmann gas

Entropy production:

$$\sigma = \partial_{\mu}S^{\mu} = \partial_{\mu}\left(-\int\frac{d^3k}{k^0}k^{\mu}f(\ln f - 1)\right) =$$

$$\frac{1}{4}\sum_{i,j,k,l}\int\frac{d^3k_i}{k_i^0}\frac{d^3k_j}{k_j^0}\frac{d^3k_k}{k_k^0}\frac{d^3k_l}{k_l^0}\left(\frac{f_k f_l}{f_i f_j} - \ln\frac{f_k f_l}{f_i f_j} - 1\right)f_i f_j W_{ij|kl} = 0$$

$$\Leftrightarrow f f_1 = f' f_1' \Leftrightarrow$$

$$f_0(x,k) = e^{\alpha(x) - \beta_{\nu}(x)k^{\nu}}$$

(local) equilibrium distribution

Thermodynamic relations - normalization

$$f_0(x,k) = e^{\alpha(x) - \beta_\nu(x) k^\nu}$$

$$N_0^\mu = \int k^\mu f_0$$

$$T_0^{\mu\nu} = \int k^\nu k^\mu f_0$$

Jüttner distribution $\quad \alpha = \frac{\mu}{T}, \beta_\nu = \frac{u_\nu}{T} \quad f_0(x,k) = e^{\frac{\mu - u_\mu k^\mu}{T}}$

$$\partial_\mu N_0^\mu = \partial_\mu \int k^\mu f_0 = \int \left(f_0 k^\mu \partial_\mu \alpha - f_0 k^\nu k^\mu \partial_\mu \beta_\nu \right) =$$

$$\boxed{\partial_\mu N_0^\mu = N_0^\mu \partial_\mu \alpha - T_0^{\mu\nu} \partial_\mu \beta_\nu}$$

$$S_0^\mu := (1 - \alpha) N_0^\mu + \beta_\nu T_0^{\mu\nu}$$

Legendre transformation and ideal gas

and $\beta_\nu = \frac{u_\nu}{T}$

$$\boxed{\partial_\mu S_0^\mu = -\alpha \partial_\mu N_0^\mu + \beta_\nu \partial_\mu T_0^{\mu\nu}}$$

Flow-frame independent thermodynamics

non-equilibrium

$$\partial_\mu S^\mu + \alpha \partial_\mu N^\mu - \beta_\nu \partial_\mu T^{\mu\nu} \geq 0$$
$$S^\mu + \alpha N^\mu - \beta_\nu T^{\mu\nu} = \Phi^\mu$$

… there are no conditions here, yet.

equilibrium

$$\partial_\mu S_0^\mu + \alpha \partial_\mu N_0^\mu - \beta_\nu \partial_\mu T_0^{\mu\nu} = 0$$
$$S_0^\mu + \alpha N_0^\mu - \beta_\nu T_0^{\mu\nu} = \beta^\mu p_0$$

$$S^\mu = s u^\mu + J^\mu$$
$$N^\mu = n u^\mu + j^\mu$$
$$T^{\mu\nu} = e u^\nu u^\mu + q^\nu u^\mu + q^\mu u^\nu + P^{\mu\nu}$$

$$u_\mu u^\mu = 1, \ \Delta^{\mu\nu} = \delta^{\mu\nu} - u^\mu u^\nu;$$
$$u_\mu J^\mu = 0, u_\mu j^\mu = 0;$$
$$u_\mu q^\mu = 0, u_\mu P^{\mu\nu} = P^{\mu\nu} u_\nu = 0.$$

$$s + \alpha n - \beta(h + w_\mu q^\mu) = 0$$
$$J^\mu + \alpha j^\mu - \beta(q^\mu + w_\nu \Pi^{\mu\nu}) + \beta p(w^\mu - g^\mu) = 0$$

$$\beta^\mu = \frac{u^\mu + w^\mu}{\sqrt{1-w^2}}, \quad \Phi^\mu = \frac{u^\mu + g^\mu}{\sqrt{1-g^2}}$$

$$S^{\mu} + \alpha N^{\mu} - \beta_{\nu} T^{\mu\nu} = \Phi^{\mu}$$

$$0 \le \partial_{\mu} S^{\mu} + \alpha \partial_{\mu} N^{\mu} - \beta_{\nu} \partial_{\mu} T^{\mu\nu} = -N^{\mu} \partial_{\mu} \alpha - T^{\mu\nu} \partial_{\mu} \beta_{\nu} + \partial_{\mu} \Phi^{\mu} =$$

$$-n\dot{\alpha} + h\dot{\beta} + q^{\mu}(\beta w_{\mu})^{\dot{}} + \beta\dot{p} + \Pi^{\mu\nu} \partial_{\mu} u_{\nu} - j^{\mu} \partial_{\mu} \alpha + \ldots = \Sigma \ge 0$$

Thermodynamics: $s(E^{\mu}, n):$

a) $ds + \alpha dn = \beta_{\mu} dE^{\mu} = \frac{1}{T}(u_{\mu} + w_{\mu}) d(eu^{\mu} + q^{\mu})$

$$w^{\mu} = 0 \Rightarrow ds + \alpha dn = \beta de$$

b) $\Phi^{\mu} = p\beta^{\mu}$ matching $S_0^{\mu} + \alpha N_0^{\mu} - \beta_{\nu} T_0^{\mu\nu} = \beta^{\mu} p_0$

$$0 \le \Sigma = \left(nw^{\mu} - j^{\mu}\right)\partial_{\mu}\alpha + \left(q^{\mu} - hw^{\mu}\right)\left(\partial_{\mu}\beta + \beta\dot{u}_{\mu}\right) + \left(\Pi^{\mu\nu} - w^{(\mu}q^{\nu)}\right)\partial_{\mu}\beta_{\nu} + w^{(\mu}q^{\nu)}\partial_{\nu}(\beta(u_{\mu} - w_{\mu}))$$

flow-frame independent

Summary of thermodynamics

Flow-EOS: $s(E^{\mu} = u_{\nu} T^{\mu\nu}, n)$

$$ds + \alpha dn = \beta_{\mu} dE^{\mu}$$

$$\beta^{\mu} = \frac{u^{\mu} + w^{\mu}}{\sqrt{1-w^2}}$$

Examples:

w^{μ}?

$$w^{\mu} = 0 \qquad w^{\mu} = \frac{q^{\mu}}{e} \qquad w^{\mu} = \frac{q^{\mu}}{h}$$

Generic stability – a benchmark

Linear asymptotic stability of homogeneous equilibrium.

Expected from non-equilibrium thermodynamics.

Second order is good, first order is bad?

- Sandovall-Villalbazzo and Garcia-Colín , et al.
  separate energy momentum balance
- Tsumura and Kunihiro
  renormalization, flow-frames
- Osada
  matching conditions
- Biró and VP
  thermodynamics

Conditions of generic stability in extended thermodynamics:

Israel–Stewart - conditional suppression (Hiscock and Lindblom, 1985):

$$S^{\mu}(T^{\mu\nu}, N^{\mu}) = \left(s(e,\, n) - \frac{\beta_0}{2T}\Pi^2 - \frac{\beta_1}{2T} q_{\nu} q^{\nu} - \frac{\beta_2}{2T}\pi^{\nu\sigma}\pi_{\nu\sigma} \right) u^{\mu} + \\ + \frac{1}{T}\left(q^{\mu} - \mu j^{\mu} + \alpha_0 \Pi q^{\mu} + \alpha_1 \pi^{\mu\nu} q_{\nu} \right)$$

$$\Omega_1 = \frac{1}{e+p}\left.\frac{\partial e}{\partial p}\right|_{\frac{s}{n}} = \frac{T}{(e+p)\left.\frac{\partial p}{\partial e}\right|_{n} - n\left.\frac{\partial p}{\partial n}\right|_{e}} \geq 0, \qquad \Omega_2 = \frac{1}{e+p}\left.\frac{\partial e}{\partial (s/n)}\right|_{p}\left.\frac{\partial p}{\partial (s/n)}\right|_{\frac{\mu}{nT}} = \ldots \geq 0,$$

Eckart frame

$$\Omega_5 = \beta_0 \geq 0, \qquad \Omega_8 = \beta_2 \geq 0, \qquad \Omega_7 = \beta_1 - \frac{\alpha_1^2}{2\beta_2} \geq 0,$$

$$\Omega_4 = e + p - \frac{2\beta_2 + \beta_1 + 2\alpha_1}{2\beta_1\beta_2 - \alpha_1^2} \geq 0, \qquad \Omega_6 = \beta_1 - \frac{\alpha_0^2}{\beta_0} - \frac{2\alpha_1^2}{3\beta_2} - \frac{1}{n^2 T}\left.\frac{\partial T}{\partial (s/n)}\right|_{n} \geq 0,$$

$$\Omega_3 = (e+p)\left(1 - \left.\frac{\partial p}{\partial e}\right|_{\frac{s}{n}} \right) - \frac{1}{\beta_0} - \frac{2}{3\beta_2} - \frac{K^2}{\Omega_6} \geq 0,$$

$$K = 1 + \frac{\alpha_0}{\beta_0} + \frac{2\alpha_1}{3\beta_2} - \frac{n}{T}\left.\frac{\partial T}{\partial n}\right|_{s/n} \geq 0.$$

Usual $\quad w^{\mu} = 0$

$$S^{\mu}(T^{\mu\nu}, N^{\mu}) = s(e, n)u^{\mu} + \beta q^{\mu} - \alpha j^{\mu}$$

$$
\begin{aligned}
\partial_{\mu} N^{\mu} &= \dot{n} + n\partial_{\mu} u^{\mu} + \partial_{\mu} j^{\mu} = 0, \\
u^{\mu}\partial_{\nu} T^{\mu\nu} &= \dot{e} + (e+p)\partial_{\mu} u^{\mu} + \partial_{\mu} q^{\mu} + q^{\mu}\dot{u}_{\mu} - \Pi^{\mu\nu}\partial_{\nu} u_{\mu} = 0, \\
\Delta^{\mu}_{\theta}\partial_{\nu} T^{\theta\nu} &= (e+p)\dot{u}^{\mu} + q^{\mu}\partial_{\nu} u^{\nu} + q^{\nu}\partial_{\nu} u^{\mu} + \Delta^{\mu}_{\theta}(\dot{q}^{\theta} + \partial_{\nu}\Pi^{\theta\nu}) = 0^{\mu}, \\
q^{\mu} &= -\lambda\Delta^{\mu\nu}\left(\partial_{\nu} T + T\dot{u}_{\nu}\right), \\
\nu^{\mu} &= -\varsigma\Delta^{\mu\nu}\partial_{\nu}\frac{\mu}{T}, \\
\Pi^{\mu}_{\ \mu} &= P^{\mu}_{\mu} - p = -\xi\partial_{\mu} u^{\mu}, \\
\Pi^{\mu}_{\ \nu} &= -2\eta < \partial_{\nu} u^{\mu} > .
\end{aligned}
$$

< > symmetric traceless spacelike part

– *generic instability*

general frame: Hiscock and Lindblom, PRD, (1985), **31**, 725.

Modified

$$w^{\mu} = \frac{q^{\mu}}{e}$$

$$S^{\mu}(T^{\mu\nu}, N^{\mu}) = s(E, n)u^{\mu} + \beta q^{\mu} - \alpha j^{\mu}$$

$$\partial_{\mu} N^{\mu} = \dot{n} + n\partial_{\mu} u^{\mu} + \partial_{\mu} j^{\mu} = 0,$$

$$u^{\mu}\partial_{\nu} T^{\mu\nu} = \dot{e} + (e+p)\partial_{\mu} u^{\mu} + \partial_{\mu} q^{\mu} + q^{\mu}\dot{u}_{\mu} - \Pi^{\mu\nu}\partial_{\nu} u_{\mu} = 0,$$

$$\Delta^{\mu}_{\theta}\partial_{\nu} T^{\theta\nu} = (e+p)\dot{u}^{\mu} + q^{\mu}\partial_{\nu} u^{\nu} + q^{\nu}\partial_{\nu} u^{\mu} + \Delta^{\mu}_{\theta}(\dot{q}^{\theta} + \partial_{\nu}\Pi^{\theta\nu}) = 0^{\mu},$$

$$q^{\mu} = -\lambda\Delta^{\mu\nu}\left(\partial_{\nu} T + T\dot{u}_{\nu} + T\frac{\dot{q}_{\nu}}{e}\right),$$

$$\nu^{\mu} = -\varsigma\Delta^{\mu\nu}\partial_{\nu}\frac{\mu}{T},$$

$$\Pi^{\mu}_{\mu} = P^{\mu}_{\mu} - p = -\xi\partial_{\mu} u^{\mu},$$

$$\Pi^{\mu}_{\nu} = -2\eta < \partial_{\nu} u^{\mu} > .$$

< > symmetric traceless spacelike part

– *generic stability*

Eckart frame: VP and Biró, TS, EPJ, (2008), **155**, 201

general: VP, J. Stat. Mech. (2009) P02054

Kinetic motivated $w^\mu = \frac{q^\mu}{h}$

$$S^\mu(T^{\mu\nu}, N^\mu) = s(e, q^\mu, n)u^\mu + \beta q^\mu - \alpha j^\mu$$

$$\begin{aligned}
\partial_\mu N^\mu &= \dot{n} + n\partial_\mu u^\mu + \partial_\mu j^\mu = 0, \\
u^\mu \partial_\nu T^{\mu\nu} &= \dot{e} + (e+p)\partial_\mu u^\mu + \partial_\mu q^\mu + q^\mu \dot{u}_\mu - \Pi^{\mu\nu}\partial_\nu u_\mu = 0, \\
\Delta^\mu_\theta \partial_\nu T^{\theta\nu} &= (e+p)\dot{u}^\mu + q^\mu \partial_\nu u^\nu + q^\nu \partial_\nu u^\mu + \Delta^\mu_\theta(\dot{q}^\theta + \partial_\nu \Pi^{\theta\nu}) = 0^\mu, \\
q^\mu &= -\lambda \Delta^{\mu\nu}\left(\partial_\nu T + T\dot{u}_\nu + \frac{T}{h}\left(\dot{q}_\mu + q_\mu \partial_\nu u^\nu + q^\nu \partial_\nu u^\mu\right)\right), \\
\nu^\mu &= -\varsigma \Delta^{\mu\nu}\partial_\nu \frac{\mu}{T}, \\
\Pi^\mu_\mu &= P^\mu_\mu - p = -\xi \partial_\mu u^\mu, \\
\Pi^\mu_\nu &= -2\eta < \partial_\nu u^\mu > .
\end{aligned}$$

< > symmetric traceless spacelike part

– *generic stability*

Eckart frame: VP and Biró, TS, PLB, (2012), **709**, 106.

Momentum density

$$T^{\mu\nu} = \begin{pmatrix} e & q^{i} \\ q^{j} & P^{ij} \end{pmatrix}$$

Not dissipative

Energy current

Summary

w^{μ} fixing is a flow-frame?

Flow related Gibbs relation:

$$ds + \mu dn = \beta_{\mu} dE^{\mu}$$

– a consequence of kinetic theory and gradient expansion
– restores generic stability of hydrodynamics
– explains temperature of moving bodies

- modified constitutive relations for $j^{\mu}, q^{\nu}, \Pi^{\mu\nu}$
- flow-frame independent entropy production


Thank you for your attention!

Eckart:

$$Tds + \mu dn = de$$

Internal energy: $\boxed{\varepsilon = e}$

$$\partial_a S^a = \dot{s}(\varepsilon, n) + s\partial_a u^a + \partial_a J^a \geq 0$$

$$J^\mu = \beta q^\mu - \alpha j^\mu$$

$$u^\mu \partial_\nu T^{\mu\nu} = \dot{e} + e\partial_\mu u^\mu + \partial_\mu q^\mu + u_\mu \dot{q}^\mu + u_\mu \partial_\nu P^{\mu\nu} = 0$$

$$\partial_\mu N^\mu = \dot{n} + n\partial_\mu u^\mu + \partial_\mu j^\mu$$

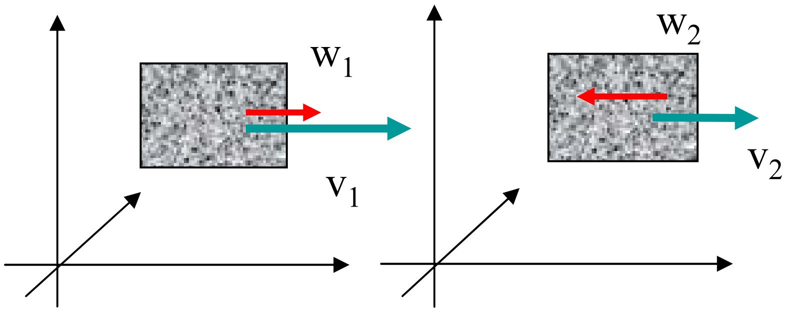
$$\left(\frac{n}{h} q^\mu - j^\mu\right)\partial_\mu \alpha + \beta \Pi^{\mu\nu}\partial_\nu u_\mu - \frac{\beta}{h} q^\mu \left(\dot{q}_\mu + q_\mu \partial_\nu u^\nu + q^\nu \partial_\nu u^\mu + \Delta^{\mu\chi}\partial_\nu \Pi_\chi{}^\nu\right) \geq 0$$

$$\boxed{\partial_\mu S^\mu + \alpha \partial_\mu N^\mu - \beta_\nu \partial_\mu T^{\mu\nu} \geq 0}$$

Benchmark 2: equilibrium and equilibration

(Einstein-Planck, Blanusa-Ott, Landsberg and Doppler)

$$TdS = g_\mu dE^\mu \quad \Rightarrow \quad \frac{g_1^\mu}{T_1} = \frac{g_2^\mu}{T_2}$$



1+1 dimension:

$$u^\mu = (\gamma, \gamma v), \quad w^\mu = (\gamma w, \gamma w)$$

$$\frac{\gamma_1(1+v_1 w_1)}{T_1} = \frac{\gamma_2(1+v_2 w_2)}{T_2}$$

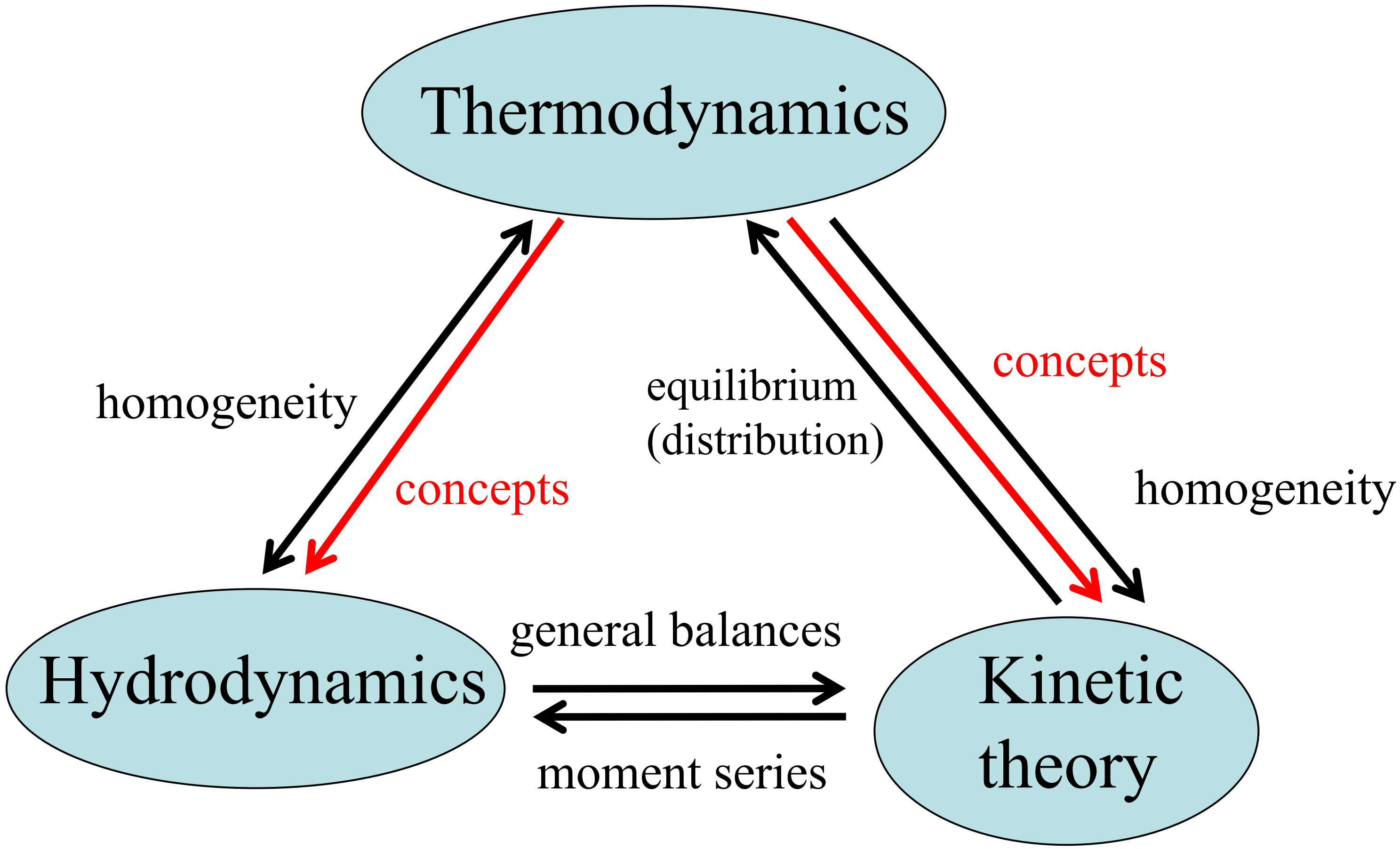
$$\frac{\gamma_1(v_1+w_1)}{T_1} = \frac{\gamma_2(v_2+w_2)}{T_2}$$

$\Rightarrow$

$$\frac{v_1+w_1}{1+v_1 w_1} = \frac{v_2+w_2}{1+v_2 w_2}$$

$$\frac{\sqrt{1-w_1^2}}{T_1} = \frac{\sqrt{1-w_2^2}}{T_2}$$

Biró, TS.-VP: EPL, **89**, 30001, 2010, (arXiv:0905.1650)



Thermodynamics
homogeneity
concepts
equilibrium
(distribution)
concepts
homogeneity
Hydrodynamics
general balances
moment series
Kinetic
theory

Expansions of different kind – entropy-wise

First order – local equilibrium (Eckart):

$$S^{\mu}(T^{\mu\nu}, N^{\mu}) = s(e,n)u^{\mu} + \frac{q^{\mu}}{T}$$

Second order - beyond local equilibrium? (Israel–Stewart):

$$S^{\mu}(T^{\mu\nu}, N^{\mu}) = \left(s(e,\ n) - \frac{\beta_0}{2T}\Pi^2 - \frac{\beta_1}{2T} q_{\nu} q^{\nu} - \frac{\beta_2}{2T} \pi^{\nu\sigma}\pi_{\nu\sigma} \right) u^{\mu} +$$

$$+ \frac{1}{T}\left(q^{\mu} + \alpha_0 \Pi q^{\mu} + \alpha_1 \pi^{\mu\nu} q_{\nu} \right)$$

+ entropy production calculation

variables: $(e, u^{\mu}, \partial_{\mu} e, \partial_{\nu} u^{\mu}, \ldots)$

functions: $(q^{\mu}, \pi^{\mu\nu}, \Pi)$

– Kinetic compatibility – a way to gradient expansion

Expansions of different kind – energy-wise

Gradient expansion in flat spacetime (Baier- et. al., 2007):

$$T^{\mu\nu} = T_0^{\mu\nu} + T_1^{\mu\nu} + T_2^{\mu\nu}$$

$$= T_0^{\mu\nu} + \pi_1^{\mu\nu}(e, u^\mu, \nabla^\mu e, \nabla^\mu u^\nu) + \pi_2^{\mu\nu}(e, u^\mu, \nabla^\mu e, \nabla^\mu u^\nu, \nabla^{\mu\nu} e, \nabla^{\mu\nu} u^\nu)$$

$$= T_0^{\mu\nu} - \eta\sigma^{\mu\nu} + \eta\tau_\pi\left(\left\langle\dot{\sigma}^{\mu\nu}\right\rangle + \frac{\partial_\lambda u^\lambda}{3}\sigma^{\mu\nu}\right) +$$

$$\lambda_2 \sigma_\lambda^{<\mu}\sigma^{\nu>\lambda} + \lambda_3 \sigma_\lambda^{<\mu}\Omega^{\nu>\lambda} + \lambda_4 \Omega_\lambda^{<\mu}\Omega^{\nu>\lambda}.$$

$$\Delta_\kappa^\nu \Delta_\lambda^\mu \partial^\lambda u^\kappa = \nabla^{\perp\nu} u^\mu = \frac{1}{2}\sigma^{\mu\nu} + \Omega^{\mu\nu} + \frac{\partial_\lambda u^\lambda}{3}\Delta^{\mu\nu}$$

Assumptions:

- equilibrium: $T_0^{\mu\nu} = e u^\mu u^\nu + \Delta^{\nu\mu} p$
- no conserved charges (Landau-Lifshitz frame)
- spacelike gradients
- covariant, isotropic, second order,
- conformal,

+ entropy production (Loganayagam, 2008)

Gradient expansion → thermodynamics

$$\partial_\nu T^{\mu\nu} = \partial_\nu (e u^\mu u^\nu + u^\mu q^\nu + u^\nu q^\mu + P^{\mu\nu}) = 0^\mu,$$

$$u^\mu \partial_\nu T^{\mu\nu} = \dot{e} + (e+p)\partial_\mu u^\mu + \partial_\mu q^\mu + q^\mu \dot{u}_\mu - \Pi^{\mu\nu}\partial_\nu u_\mu = 0,$$

$$\Delta^\mu_\theta \partial_\nu T^{\theta\nu} = (e+p)\dot{u}^\mu + q^\mu \partial_\nu u^\nu + q^\nu \partial_\nu u^\mu + \Delta^\mu_\theta (\dot{q}^\theta + \partial_\nu \Pi^{\theta\nu}) = 0^\mu,$$

$$\partial_\mu S^\mu = \partial_\mu (s u^\mu + J^\mu) \geq 0.$$

– Variables (first order expansion): $(e, u^\mu, \partial_\mu e, \partial_\nu u^\mu)$
– Functions: $(q^\mu, \Pi^{\mu\nu}, s, J^\mu)$

$$\partial_\mu S^\mu - \beta_\mu \partial_\nu T^{\mu\nu} \geq 0$$

$\beta_\mu = \beta(u_\mu + w_\mu)$ reciprocal temperature (vector)

$$\partial_{\mu} S^{\mu} - \beta_{\mu} \partial_{\mu} T^{\mu\nu} \left(- \lambda \partial_{\mu} N^{\mu} \right) \geq 0$$

Condition:

Entropy production is independent of acceleration.

Consequences:

1) $$s(e,u^{a},n) = s(e,q^{a}(e,u^{a}),n)$$

2) $$e \frac{\partial s}{\partial q^{a}} = q_{a} \frac{\partial s}{\partial e} \Rightarrow s(e,q^{a}) = \hat{s}(e^{2} - \mathbf{q}^{2}) = \tilde{s}\left(\sqrt{e^{2} - \mathbf{q}^{2}}\right) = \tilde{s}\left(\left\| E^{\mu} \right\|\right)$$

$$E^{\mu} = eu^{\mu} + q^{\mu}, \qquad w^{\mu} = \frac{q^{\mu}}{e}$$

3) $$J^{\mu} = \beta \left(q^{\mu} - \frac{q_{\nu}}{e} P^{\mu\nu} \right)$$

JMMS, 2008, **3**/6, 1161, (arXiv:07121437)

Summary of gradient expansion equilibrium:

– Gibbs relation (of Israel) = no entropy production:

$$\partial_\mu S^\mu - \beta_\mu \partial_\nu T^{\mu\nu} = 0 \quad \Rightarrow \quad \frac{\partial s}{\partial E^\mu} = \beta_\mu$$

$$\boxed{Tds = g_\mu dE^\mu} = (u_\mu + w_\mu) d(eu^\mu + q^\mu)$$

– Preferable simplification

$$\boxed{Tds + \mu dn = (u_\mu + w_\mu) dE^\mu = de + w_\mu dq^\mu + (ew_\mu - q_\mu) du^\mu = de + \frac{q_\mu}{e} dq^\mu}$$