GR on the Computer

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Compact object zoo

- End products of stellar evolution (compact stars)
  - White dwarfs ($\sim r_{\text{Earth}}, \sim M_\odot$) remnants of stellar collapse ($< 8M_\odot$)
  - Neutron stars ($r_{\text{NS}} \sim 10\text{km}, \lesssim 2.17 M_\odot \equiv M_{\text{TOV}}, \rho_c \geq \rho_0$); produced by supernovae ($8M_\odot$ to $\sim 25M_\odot$), upheld by neutron degeneracy pressure
  - Stellar black holes, equilibrium of degeneracy pressure against gravity breaks down, escape velocity reaches $c$; created by stars $\geq 25M_\odot$

- Other compact objects
  - Intermediate and supermassive black holes
  - Exotic stars (Quark stars, Boson stars, etc.)
Solving General Relativity

- **GR** solutions depend on the considered system
  - with enough symmetries → **analytical** solutions
  - with small parameters → **pertubative** solutions
  - If no symmetries, strong fields and dynamic system → solutions by **numerical relativity**
  - Evolution of the system → **relativistic hydrodynamics**

- **Equation of state** determines solutions: \( P(\rho) \)
- Exact **EOS** for dense matter is not known → different models in use

- **Mass radius relation**, causality and (future) observations constraint **EOS**
Numerical Relativity and Relativistic Hydrodynamics of Binary Neutron Star Mergers

A realistic numerical simulation of a twin star collapse, a merger of two compact stars or a collapse to a black hole needs to go beyond a static, spherically symmetric TOV-solution of the Einstein- and hydrodynamical equations.

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}, \]
\[ \nabla_{\mu} (\rho u^{\mu}) = 0, \]
\[ \nabla_{\nu} T^{\mu\nu} = 0. \]

(3+1) decomposition of spacetime

\[ g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i \beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix} \]

\[ d\tau^2 = \alpha^2(t, x^j) dt^2 \]
\[ x^i_{t+dt} = x^i_t - \beta^i(t, x^j) dt \]

Numerical relativity 1/4

- Evolve spacetime: 3+1 split $\rightarrow$ back to space+time
- Define spacelike hyper surfaces $\Sigma_t$ with:
  - Normal one form $\Omega_\mu = \nabla_\mu t$ and...
  - Unit normals: $n_\mu = -\alpha \Omega_\mu$ (future pointing)
- Spatial tensor contractions vanish: $v^\mu n_\mu = 0$
- Differential geometry in $\Sigma_t$ (metric, derivative ...):
  - Metric: $\gamma_{\rho\sigma} = g_{\alpha\beta} \gamma^\alpha_\sigma \gamma^\beta_\rho = g_{\alpha\beta} + n_\alpha n_\beta$
  - Derivative: $D_\alpha f = \gamma^{\beta}_\alpha \nabla_\beta f$ and $D_\alpha v^\beta = \gamma^{\sigma}_\alpha \gamma^\beta_\rho \nabla_\sigma v^\rho$
  - Spatial Riemann tensor: $(^3)R^\rho_{\sigma\alpha\beta} v_\rho = 2D_{[\beta} D_\alpha] v_\rho$
Numerical relativity 2/4

- Embed $\Sigma_t$ in the manifold with extrinsic curvature:

$$K_{\sigma\rho} = \gamma^\alpha_{\sigma} \gamma^\beta_{\rho} \nabla(\alpha n_\beta) = -1/2 \mathcal{L}_n \gamma_{\sigma\rho}$$

- Relate 4D and 3D curvature: Gauss, Cadazzi and Ricci equations (plenty calculations!)

- Adopt coordinates $x^\alpha = [t, x^i]$ and time vector $t^\sigma = \alpha n^\sigma + \beta^\sigma$

- Decompose Einstein to 3+1 equations ($S_{\sigma\rho} = \gamma^\alpha_{\sigma} \gamma^\beta_{\rho} T_{\alpha\beta}$):

  - $nn$: $(^3)R + K^2 - K_{ij} K^{ij} = 16\pi \rho$ \hspace{1cm} ($\rho = T_{\alpha\beta} n^\alpha n^\beta$) Hamiltonian constraint (H)
  
  - $n\gamma$: $D_j (K^{ij} - \gamma^{ij} K) = 8\pi S^i$ \hspace{1cm} Momentum constraint (M)

  $$\gamma\gamma: \partial_t K_{ij} = \alpha (^3R_{ij} - 2K_{ik} K^k_j + KK_{ij} - D_i D_j \alpha - 8\pi \alpha (S_{ij} - 1/2 \gamma_{ij} (S - \rho)))$$

  $$+ \beta^k \partial_k K_{ij} + K_{ik} \partial_j \beta^k + K_{kj} \partial_i \beta^k$$

  - Definition of $K_{ij}$: $\partial_t \gamma_{ij} = - 2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$
Numerical relativity 3/4

- Coordinate freedom by $\alpha$, $\beta$:
  
  \[ ds^2 = -\alpha \, dt^2 + \gamma_{ij} \left( dx_i + \beta_i dt \right) \left( dx_j + \beta_j dt \right) \]

- So far so good... but...
  
  - Solve constraints for $\gamma_{ij}$, $K_{ij}$ (H) and (M) are of no known math type
  
  - Choose gauge $\alpha$, $\beta$ Singularities have to be avoided, have to be chosen before solution
  
  - Evolve $\partial_t \gamma_{ij}$, $\partial_t K_{ij}$ Not well posed equations! Numerics will fail.

- Solution: conformal transformation: $\gamma_{ij}(x^i) = \phi^4(x^i) \gamma_{ij}^\sim(x^i)$
  
  - $\phi(x^i) > 0$; induces conformal geometry $(\mathcal{R}^\sim, D^\sim_i, \ldots)$
Numerical relativity 4/4: conclusion

• Further steps:
  – Conformal trafo yields elliptic equation for \( (H') \)
  – Transverse (divergence free), traceless decomposition of \( K_{ij} = A_{ij} + \frac{1}{3} \gamma_{ij} K \) yields elliptic operator for \( A_{ij} = \phi^{-10} \tilde{A}_{ij} \)
  – Trafo of equations yields \textbf{4 coupled, elliptic 2\textsuperscript{nd} order PDEs}, can be solved numerically!
  – Solvable with boundary conditions (problematic for BH with spin >0.9)
• Evolution of initial data: e.g. BSSNOK equations
  – \( \partial_t \gamma^*_{ij} = \ldots; \partial_t \tilde{A}_{ij} = \ldots; \partial_t \phi = \ldots; \partial_t K = \ldots; \partial_t \Gamma_i = \ldots \) (see [1])
• Alternative formulations: CCZ4, generalized harmonic formulation
The ADM equations

The ADM (Arnowitt, Deser, Misner) equations come from a reformulation of the Einstein equation using the (3+1) decomposition of spacetime.

\[ \partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij} \]
\[ = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \]

\[ \partial_t K_{ij} = -D_i D_j \alpha + \beta^k \partial_k K_{ij} + K_{ik} \partial_j \beta^k + K_{kj} \partial_i \beta^k + \alpha \left( R_{ij} + K K_{ij} - 2K_{ik} K^k_j \right) + 4\pi \alpha \left[ \gamma_{ij} (S - E) - 2S_{ij} \right] \]

\[ D_j \left( K^{ij} - \gamma^{ij} K \right) = 8\pi S^i \]

\[ (3) R + K^2 - K_{ij} K^{ij} = 16\pi E \]

Three dimensional covariant derivative

\[ D_\nu := \gamma^\mu_\nu \nabla_\mu = (\delta^\mu_\nu + n_\nu n^\mu) \nabla_\mu \]

Extrinsic Curvature:

\[ K_{\mu\nu} := -\gamma^\lambda_\mu \nabla_\lambda n_\nu \]

Three dimensional Riemann tensor

\[ (3) R^{\mu}_{\nu\kappa\sigma} = \partial_\kappa (3) \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma (3) \Gamma^{\mu}_{\nu\kappa} + (3) \Gamma^{\lambda}_{\nu\kappa} (3) \Gamma^\lambda_\nu_\sigma - (3) \Gamma^\lambda_\nu_\sigma (3) \Gamma^\lambda_\nu_\kappa \]

\[ (3) \Gamma^\alpha_{\beta\gamma} = \frac{1}{2} \gamma^\alpha_\delta \left( \partial_\beta \gamma^\gamma_\delta + \partial_\gamma \gamma^\delta_\beta - \partial^\delta \gamma^\beta_\gamma \right) \]

From ADM to BSSNOK

Unfortunately the ADM equations are only weakly hyperbolic (mixed derivatives in the three dimensional Ricci tensor) and therefore not "well posed". It can be shown that by using a conformal traceless transformation, the ADM equations can be written in a hyperbolic form. This reformulation of the ADM equations is known as the BSSNOK (Baumgarte, Shapiro, Shibata, Nakamuro, Oohara, Kojima) formulation of the Einstein equation. Most of the numerical codes use this (or even better the CCZ4) formulation.

The 3+1 Valencia Formulation of the Relativistic Hydrodynamic Equations

To guarantee that the numerical solution of the hydrodynamical equations (the conservation of rest mass and energy-momentum) converge to the right solution, they need to be reformulated into a conservative formulation. Most of the numerical “hydro codes” use the 3+1 Valencia formulation.
Finite difference methods

Discretisation of a hyperbolic initial value boundary problem.

High resolution shock capturing methods (HRSC methods are needed, when Riemann problems of discontinuous properties and shocks needs to be evolved accurately.)

Gauge Conditions

On each spatial hypersurface, four additional degrees of freedom need to be specified: A slicing condition for the lapse function and a spatial shift condition for the shift vector need to be formulated to close the system. In an optimal gauge condition, singularities should be avoided and numerical calculations should be less time consuming.

Bona-Massó family of slicing conditions:

“1+log” slicing condition: $f = 2/\alpha$

where $f(\alpha) > 0$ and $K_0 := K(t = 0)$

“Gamma-Driver” shift condition:

$\partial_t \alpha - \beta^k \partial_k \alpha = -f(\alpha) \alpha^2 (K - K_0)$

$\partial_t \beta^i - \beta^j \partial_j \beta^i = \frac{3}{4} B^i$

$\partial_t B^i - \beta^j \partial_j B^i = \partial_t \tilde{\Gamma}^i - \beta^j \partial_j \tilde{\Gamma}^i - \eta B^i$
Twin stars

- Solutions for **NS** with two equilibria

**FIG. 2.** Stellar sequence of neutron stars and their twins as a function of central density. Both segments with positive slope correspond to stable configurations since their normal modes of vibration are found to be stable.
Teil III

Inhalte des Teil III:

• How to download and build (compile) the Einstein Toolkit
• How to run a test simulation (static_tov.par)
• Run and visualize (Mathematica or Python) one of the following problems
  • Migration of an unstable neutron star to a stable configuration
  • Collapse of an unstable neutron star to a black hole
  • Collapse of a neutron star to a quark star (twin star collapse)
Software: Component Framework

- Domain Scientists
- Computational Relativists
- CDSE
- CS

Groups:
- Group A Thorns
- Group B Thorns

Toolkits:
- Einstein Toolkit
- Cactus Computational Toolkit

Platforms:
- Cactus Flesh (APIs and Definitions)
- Driver Thorns (Parallelisation)

Languages:
- MPI, Threads, New Programming Models
Einstein Toolkit

“The Einstein Toolkit Consortium is developing and supporting open software for relativistic astrophysics. Our aim is to provide the core computational tools that can enable new science, broaden our community, facilitate interdisciplinary research and take advantage of emerging petascale computers and advanced cyberinfrastructure.”

- Consortium: 94 members, 49 sites, 14 countries
- Sustainable community model:
  - 9 Maintainers from 6 sites: oversee technical developments, quality control, verification and validation, distributions and releases
  - Whole consortium engaged in directions, support, development
  - Open development meetings
  - Governance model: still being discussed (looking at CIG, iPlant)

HTTP://WWW.EINSTEINTOOLKIT.ORG
Das Einstein Toolkit

Einstein Toolkit School and Workshop

Join us at the North American Einstein Toolkit School and Workshop at NCSA, at the University of Illinois at Urbana-Champaign from July 31 to August 4, 2017.

This meeting is open to anyone interested in numerical relativity and computational astrophysics and cosmology and in particular to Einstein toolkit users.

The first three days will be dedicated to a school useful for new users of the Einstein Toolkit followed by a two-day long workshop open to developers interested in the Einstein Toolkit.

Registration closes July 17, 2017.

About

The Einstein Toolkit is a community-driven software platform of core computational tools to advance and support research in relativistic astrophysics and gravitational physics.

Download

We provide a convenient method to get all of the Einstein Toolkit with just a few commands, and explain the whole process.

Documentation

A lot of the documentation within the Einstein Toolkit is generated from comments in the source code, and more can be found on the Einstein Toolkit Wiki or other documents. We provide links to guides, tutorials and references.

Contribute

The Einstein Toolkit would not exist without numerous contributions from its community. It is easy to learn how you can contribute as well.
Das Einstein Toolkit: Download

Download & Requirements

The Einstein Toolkit is hosted on many different machines around the world. We provide a script called `GetComponents` to simplify downloading the toolkit. This page just describes how to download the toolkit - you may also be interested in the Tutorial for New Users which leads you through these steps and more on the Queen Bee supercomputer, or in a simpler tutorial for setup on a typical Linux box.

Users of the Einstein Toolkit are encouraged to register which also signs up for the users mailing list.

Main Toolkit

Citations

The development of production level scientific software, such as the components of the Einstein Toolkit, represents the academic output of researchers. These scientific contributions should be acknowledged and respected on par with those solely based in theory or experiment. Please review our Citation Policy.

Current release: Payne-Gaposchkin (released on December 16th, 2016)

This is the recommended version of the toolkit for most users. See the release notes for more information.

Note: OSX users cannot use the 'subversion' client shipped by Apple. In that case install subversion either from homebrew or macports.

Enter the directory on your machine in which you would like to download the ET (for example, your home directory), and type the commands listed below. This will create a directory called Cactus in which the components of the Einstein Toolkit are downloaded.

```
```

A tarball of the release is also available here, but using `GetComponents` is the preferred method to obtain the code. Use the tarball only if there is no way to use `GetComponents` (which should almost never be the case).
ET-Download auf dem Fuchs-Cluster

Checking out module: par
from repository: https://bitbucket.org/einstein_toolkit/einsteinexamples.git
into: Cactus


Checking out module: Dependencies1
from repository: https://bitbucket.org/einstein_toolkit/cactus_dep2.git
into: Cactus/arrangements


Checking out module: Dependencies2
from repository: https://bitbucket.org/einstein_toolkit/cactus_test.git
into: Cactus/arrangements


Checking out module: CactusTest/TestFortran
from repository: https://bitbucket.org/cactus/cactus_test.git
into: Cactus/arrangements


Checking out module: QuasiLocalMeasures
from repository: https://bitbucket.org/einstein_toolkit/einsteinanalysis.git
into: Cactus/arrangements


Checking out module: InitialData/IodThird
from repository: https://bitbucket.org/einstein_toolkit/einsteininitialdata.git
into: Cactus/arrangements


Checking out module: PITTNullCode/SphericalHarmonicRecon
from repository: https://bitbucket.org/einstein_toolkit/pitt_nullcode.git
into: Cactus/arrangements


Checking out module: PITTNullCode/SphericalHarmonicDecomp
from repository: https://bitbucket.org/einstein_toolkit/pitt_nullcode.git
into: Cactus/arrangements


Checking out module: EinsteinExact/ModifiedSpherical
from repository: https://github.com/barrywardell/einsteinexact.git
into: Cactus/arrangements


Checking out module: EinsteinInitialData/Hydro_Initial
from repository: https://bitbucket.org/einstein_toolkit/einsteininitialdata.git
into: Cactus/arrangements


Checking out module: EinsteinUtils/GetMask_SphericalSurface
from repository: https://bitbucket.org/einstein_toolkit/einsteinutils.git
into: Cactus/arrangements


Checking out module: PITTNullCode/SphericalHarmonicReconGen
from repository: https://bitbucket.org/einstein_toolkit/pitt_nullcode.git
into: Cactus/arrangements


Checking out module: EinsteinInitialData/IDConstraintViolate
from repository: https://bitbucket.org/einstein_toolkit/einsteininitialdata.git
into: Cactus/arrangements


Checking out module: CONTRIBUTORS
from repository: https://bitbucket.org/cactuscode/cactus.git
into: Cactus

as: flesh

Checking out module: COPYRIGHT
from repository: https://bitbucket.org/cactuscode/cactus.git
into: Cactus

as: flesh

Checking out module: doc
from repository: https://bitbucket.org/cactuscode/cactus.git
into: Cactus

as: flesh

Checking out module: lib
from repository: https://bitbucket.org/cactuscode/cactus.git
into: Cactus

as: flesh
Das Einstein Toolkit: Setup mit SimFactory

[prakti@login02.csc Cactus]$ ./simfactory/bin/sim setup --machine fuchs

Here we will define some necessary Simulation Factory defaults.

Determining local machine name: login02.cm.cluster
Creating machine login02.cm.cluster from generic: machine login02.cm.cluster [/home/agmisc/prakti/ET-2016-11/Cactus/repos/simfactory2/mdb/machine]

enter value for key user [prakti]:
enter value for key email [prakti]:
enter value for key allocation []:
enter value for key sourcebasedir (the parent directory containing the Cactus sourcetree) [/home/agmisc/prakti/ET-2016-11]:
enter value for key basedir (the location of simfactory simulations) [/home/agmisc/prakti/simulations]:

Would you like to enter key/value pairs for a specific machine? [Y/N*]:

------------------------SUMMARY------------------------:

[default]
user  = prakti
email = prakti
allocation =
sourcebasedir = /home/agmisc/prakti/ET-2016-11
basedir = /home/agmisc/prakti/simulations

------------------------END SUMMARY------------------------:

Save contents [Y*/N*]:
Contents successfully written to /home/agmisc/prakti/ET-2016-11/Cactus/repos/simfactory2/etc/defs.local.ini
[prakti@login02.csc Cactus]$
Das Einstein Toolkit: Kompilierung

```
(prakt1@login02.csc Cactus)[$ ./simfactory/bin/sim build et --thornlist ./manifest/einsteintoolkit.th --machine fuchs
Using configuration: et
Reconfiguring et
Writing configuration to: /home/agmisc/prakt1/ET-2016-11/Cactus/configs/et/OptionList
Cactus - version: 4.2.3
Reconfiguring et.
Using configuration options from configure line
    Setting fds to '4.5 -j --'
End of options from configure line
Adding configuration options from '/home/agmisc/prakt1/ET-2016-11/Cactus/configs/et/OptionList'...
    Setting CPP to 'cpp'
    Setting FPP to 'cpp'
    Setting CC to '/cm/shared/apps/intel/compiler/xe/2013_sp1.3.174/compiler_xe_2013_sp1.3.174/bin/intel64/icc'
    Setting CXX to '/cm/shared/apps/intel/compiler/xe/2013_sp1.3.174/compiler_xe_2013_sp1.3.174/bin/intel64/icpc'
    Setting F77 to '/cm/shared/apps/intel/compiler/xe/2013_sp1.3.174/compiler_xe_2013_sp1.3.174/bin/intel64/ifort'
    Setting F90 to '/cm/shared/apps/intel/compiler/xe/2013_sp1.3.174/compiler_xe_2013_sp1.3.174/bin/intel64/ifort'
    Setting CPPFLAGS to '-DCCTK_DISABLE_OMP_COLLAPSE -DCCTK_DISABLE_RESTRICT'
    Setting FPPFLAGS to '-DCCTK_DISABLE_OMP_COLLAPSE -traditional -DCCTK_DISABLE_RESTRICT'
    Setting CFLAGS to '-g -traceback -msse3 -align -std=c99 -U STRICT Ansi'
    Setting CXXFLAGS to '-g -traceback -msse3 -align -std=c++11 -D __builtin_fmaxf=fmaxf -D __builtin_fmaxl=fmaxl -D __builtin_fminf=fminf -D __builtin_fminl=fminl'
    Setting F77FLAGS to '-g -traceback -msse3 -align -pad -safe-cray-ptr'
    Setting F90FLAGS to '-g -traceback -msse3 -align -pad -safe-cray-ptr'
    Setting C_LINE_DIRECTIVES to 'yes'
    Setting F_LINE_DIRECTIVES to 'yes'
    Setting BEGIN_WHOLE_ARCHIVE_FLAGS to ' -Wl,--whole-archive'
    Setting END_WHOLE_ARCHIVE_FLAGS to ' -Wl,--no-whole-archive'
    Setting VECTORISE to 'yes'
    Setting VECTORISE_ALIGNED ARRAYS to 'no'
    Setting VECTORISE_INLINE to 'no'
    Setting VECTORISE_STREAMING_STORES to 'no'
    Setting DEBUG to 'no'
    Setting CPP_DEBUG_FLAGS to '-DCARPET_DEBUG'
    Setting FPP_DEBUG_FLAGS to '-DCARPET_DEBUG'
    Setting C_DEBUG_FLAGS to '-00 -debug all'
    Setting CXX_DEBUG_FLAGS to '-00 -debug all'
    Setting F77_DEBUG_FLAGS to '-00 -debug all'
    Setting F90_DEBUG_FLAGS to '-00 -debug all'
```
Welcome

The Einstein Toolkit Consortium is developing and supporting open software for relativistic astrophysics. Our aim is to provide the core computational tools that can enable new science, broaden our community, facilitate interdisciplinary research and take advantage of emerging petascale computers and advanced cyberinfrastructure.

Please read our pages about the Einstein Toolkit, its governance, and how to get started with the toolkit for more information.

Download

November 2014: We are pleased to announce the tenth release (code name "Herschel") of the Einstein Toolkit, an open, community developed software infrastructure for relativistic astrophysics.

https://www.youtube.com/watch?v=EO4d32ch6OI
https://www.youtube.com/watch?v=p5bq2iUO3DE
https://www.youtube.com/watch?v=MNpyd_o0MT4
https://www.youtube.com/watch?v=Qg6PwRI2uS8
https://www.youtube.com/watch?v=ZW3aV7U-aik
Equation of State examples

\[ \rho / \rho_0 \]

\[ P \text{[MeVfm}^{-3}] \]

- FSU2H
- FSU2H new
- EOS Blaschke

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15. June 2018
FIG. 6. FSU2Hnew and Blaschke equation of states density to pressure and speed of sound profile.
FIG. 7. Comparison of mass to radius relation of different equations of state.
FSU2H Twin star oscillations

\[ t = 0.59 \text{ ms} \quad 0.99 \text{ ms} \quad 1.38 \text{ ms} \]

\[ \rho_{\text{max}} / \rho_{\text{nuc}} \]

\[ \log(\rho) [\text{g/cm}^3] \]

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15. June 2018
References


