Towards the equation of state of hot QCD at finite baryon density

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QCD phase diagram: towards finite density

\[ \mu_B = 0 \quad \rightarrow \quad T - \mu_B \text{ plane} \]

- QCD EoS at \( \mu_B = 0 \) available from first-principle lattice QCD simulations
- QCD EoS at finite density necessary for many applications, including hydro modeling of heavy-ion collisions at RHIC, SPS, FAIR energies
- Implementation of the QCD critical point necessary to look for its signatures
Non-zero $\mu_B$ and lattice QCD

At $\mu_B \neq 0$ fermion determinant is complex: $\det M[U, \mu] = |\det M[U, \mu]| e^{i\theta}$

“Probability distribution” interpretation is lost $\rightarrow$ lattice method inapplicable

Indirect lattice methods:

- Taylor expansion

$$\frac{p(T, \mu_B)}{T^4} = \frac{p(T, 0)}{T^4} + \frac{\chi^B_2(T, 0)}{2!}(\mu_B/T)^2 + \frac{\chi^B_4(T, 0)}{4!}(\mu_B/T)^4 + \ldots$$

$\chi^B_k$ – cumulants (susceptibilities) of net baryon distribution

Can be computed in Lattice QCD at $\mu_B = 0$

- Analytic continuation from imaginary $\mu_B$

No sign problem at $\mu_B = i\mu_B$

Compute at $\mu_B^2 < 0$ and continue to $\mu_B^2 > 0$

All these lattice methods inherently limited to “small” $\mu_B$

A more practical approach: use lattice data to constrain effective models

[Wuppertal-Budapest collaboration, 1607.02493]
Outline

1. Taylor expansion from lattice QCD
   • Model-independent method with a limited scope (small $\mu_B/T$)
   • State-of-the-art and estimates for radius of convergence

2. Lattice-based effective models
   • Cluster expansion model (CEM)
   • Hagedorn bag-like model
   • Chiral mean-field model

3. Status of the critical point at finite density
Finite $\mu_B$ EoS from Taylor expansion

\[
\frac{p(T, \mu_B)}{T^4} = \frac{p(T, 0)}{T^4} + \frac{\chi^B_2(T, 0)}{2!} (\mu_B/T)^2 + \frac{\chi^B_4(T, 0)}{4!} (\mu_B/T)^4 + \ldots
\]

$\chi^B_k$ – cumulants of net baryon distribution, computed up to $\chi^B_8$

- Off-diagonal susceptibilities also available → incorporate conservation laws $n_S = 0, n_Q/n_B = 0.4$
- Method inherently limited to “small” $\mu_B/T$, within convergence radius

[HotQCD collaboration, 1701.04325] [Wuppertal-Budapest collaboration, 1805.04445]
Taylor expansion and radius of convergence

A truncated Taylor expansion only useful within the radius of convergence. Its value is a priori unknown. Any singularity in complex $\mu_B$ plane will limit the convergence, it does not have to be a phase transition or a critical point.
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**An example:** HRG model with a baryonic excluded volume (EV)

$V \rightarrow V - bN$

$\lambda_B = e^{\mu_B/T}$

Lambert $W(z)$ function has a branch cut singularity at $z = -e^{-1}$, corresponds to a negative (unphysical) fugacity.

[Taradiy, Motornenko, V.V., Gorenstein, Stoecker, 1904.08259]
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The best one can do with Taylor expansion

Truncated LQCD Taylor expansion

\[
\frac{p}{T^4} = \sum_{i,j,k} \frac{\chi_{i,j,k}^{BQS}(T)}{i!j!k!} \left( \frac{\mu_B}{T} \right)^i \left( \frac{\mu_Q}{T} \right)^j \left( \frac{\mu_S}{T} \right)^k
\]

HRG model at smaller temperatures

\[
\frac{p}{T^4} = \sum_{i \in \text{hrg}} T \phi_i^{id}(T) e^{\mu_B/T} e^{\mu_Q/T} e^{\mu_S/T}
\]

- Includes the three conserved charges and conservation laws, no criticality
- Probably best one can do with Taylor expansion. Applications: RHIC BES

[Monnai, Schenke, Shen, 1902.05095]

[Noronha-Hostler, Parotto, Ratti, Stafford, 1902.06723]
Truncated Taylor expansion and imaginary $\mu_B$

Are we using all information available from lattice? Consider relativistic virial expansion (Laurent series in fugacity $p = \sum_{k=-\infty}^{\infty} p_k \ e^{k \mu_B / T}$) and imaginary $\mu_B$

\[
\left. \frac{\rho_B}{T^3} \right|_{\mu_B = i \theta_B T} = i \sum_{k=1}^{\infty} b_k(T) \sin(k \theta_B) \quad \Rightarrow \quad b_k(T) = -\frac{2i}{\pi} \int_0^{\pi} \frac{\rho_B(T, i \theta_B T)}{T^3} \sin(k \theta_B) \ d\theta_B
\]

Relativistic virial/cluster expansion \quad Fourier coefficients
Truncated Taylor expansion and imaginary $\mu_B$

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Relativistic virial/cluster expansion

**Lines:** Taylor expansion up to $\chi_B^4$ using lattice data, as in 1902.06723

**Symbols:** Lattice data for $b_k$ from imaginary $\mu_B$

[V.V., Pasztor, Fodor, Katz, Stoecker, 1708.02852]

Quite some room for improvement at $T<200$ MeV
Cluster Expansion Model — CEM

a model for QCD equation of state at finite baryon density constrained to both susceptibilities and Fourier coefficients

Cluster Expansion Model (CEM)

Model formulation:

- Relativistic virial (cluster) expansion for baryon number density
  \[
  \frac{\rho_B(T, \mu_B)}{T^3} = \chi_1^B(T, \mu_B) = \sum_{k=1}^{\infty} b_k(T) \sinh(k\mu_B/T)
  \]

- \(b_1(T)\) and \(b_2(T)\) are model input from lattice QCD

- All higher order coefficients are predicted: \(b_k(T) = \alpha_k^{SB} \left[ b_2(T) \right]^{k-1} / \left[ b_1(T) \right]^{k-2}\)

Physical picture: Hadron gas with repulsion at moderate \(T\), QGP-like phase at high \(T\)

Summed analytic form:

\[
\frac{\rho_B(T, \mu_B)}{T^3} = -\frac{2}{27\pi^2} \frac{\hat{b}_1^2}{\hat{b}_2} \left\{ 4\pi^2 \left[ \text{Li}_1(x_+) - \text{Li}_1(x_-) \right] + 3 \left[ \text{Li}_3(x_+) - \text{Li}_3(x_-) \right] \right\}
\]

\[
\hat{b}_k = \frac{b_k(T)}{b_k^{SB}}, \quad x_\pm = -\frac{\hat{b}_2}{\hat{b}_1} e^{\pm\mu_B/T}, \quad \text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}
\]

Regular behavior at real \(\mu_B\) \(\rightarrow\) no-critical-point scenario
CEM: Fourier coefficients

\[ b_k(T) \]

\[ \alpha_k(T) \equiv b_k(T) \frac{[b_1(T)]^{k-2}}{[b_2(T)]^{k-1}} \]

CEM: \( b_1(T) \) and \( b_2(T) \) as input \( \rightarrow \) consistent description of \( b_3(T) \) and \( b_4(T) \)

Lattice data on \( b_{3,4}(T) \) inconclusive at \( T \leq 170 \, \text{MeV} \)
CEM: Baryon number susceptibilities

\[ \chi_k^B (T, \mu_B) = -\frac{2}{27\pi^2} \frac{\hat{b}_1^2}{b_2} \left\{ 4\pi^2 \left[ \text{Li}_{2-k} (x_+) + (-1)^k \text{Li}_{2-k} (x_-) \right] + 3 \left[ \text{Li}_{4-k} (x_+) + (-1)^k \text{Li}_{4-k} (x_-) \right] \right\} \]

Lattice data from 1805.04445 (Wuppertal-Budapest), 1701.04325 & 1708.04897 (HotQCD)
CEM: Equation of state

\[
\frac{p(T, \mu_B)}{T^4} = \frac{p_0(T)}{2} - \frac{2}{27\pi^2} \frac{b_1^2}{b_2} \left\{ 4\pi^2 \left[ \text{Li}_2(x_+) - \text{Li}_2(x_-) \right] + 3 \left[ \text{Li}_4(x_+) - \text{Li}_4(x_-) \right] \right\}
\]

**Input:** \( p_0(T), b_{1,2}(T) \) ← parametrized LQCD + HRG

Tabulated CEM EoS available at https://fias.uni-frankfurt.de/~vovchenko/cem_table/

Currently restricted to single chemical potential (\( \mu_B \)) and no critical point
Hagedorn (bag-like) resonance gas model with repulsive interactions
exactly solvable model with a (phase) transition between hadronic matter and QGP

[Gorenstein, Petrov, Zinovjev, PLB '81; Gorenstein, W. Greiner, Yang, JPG '98; Ferroni, Koch, PRC '09]

Here the model equation of state is constrained to lattice QCD

Hagedorn bag-like model: formulation

- HRG + quark-gluon bags: $\rho_Q(m, \nu) = C \nu^\gamma (m - B\nu)^\delta \exp \left\{ \frac{4}{3} [\sigma_Q]^{1/4} \nu^{1/4} (m - B\nu)^{3/4} \right\}$
- Non-overlapping particles (excluded volume correction): $V \to V - bN$
- Isobaric (pressure) ensemble: $(T, V, \mu) \to (T, s, \mu)$
- Massive (thermal) partons (new element)
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Resulting picture: transition (crossover, 1st order, 2nd order, etc.) between HRG and MIT bag model EoS, within single partition function

"Crossover" parameter set

\( \gamma = 0, \quad \delta = -2, \quad C = 0.03, \quad V_0 = 8 \text{ fm}^3 \)
\( m_u = m_d = 300 \text{ MeV}, \quad m_s = 350 \text{ MeV} \)
\( m_g = 800 \text{ MeV}, \quad B^{1/4} = 200 \text{ MeV} \)

\( T_H \simeq 167 \text{ MeV} \)
Hagedorn model: Susceptibilities

Lattice data from 1112.4416 (Wuppertal-Budapest), 1203.0784 (HotQCD)
Hagedorn model: Susceptibilities and Fourier

\[ \chi_4^B / \chi_2^B \]

Massive quarks
\[ B^{1/4} = 200 \text{ MeV} \]
\[ \gamma = 0, \delta = -2 \]
\[ V_0 = 8 \text{ fm}^3 \]

\[ \chi_6^B / \chi_2^B \]

\[ \chi_8^B \]

\[ b_k \]
Hagedorn model: Finite baryon density

- Crossover transition to a QGP-like phase in both the $T$ and $\mu_B$ directions
- Essentially a built-in “switching” function between HRG and QGP, thermodynamically consistent by construction (single partition function)
- Critical point/phase transition at finite $\mu_B$ can be incorporated through $\mu_B$-dependence of $\gamma$ and $\delta$ exponents in bag spectrum

see Gorenstein, Gazdzicki, Greiner, Phys. Rev. C (2005)

More on this at SQM2019
SU(3) parity-doublet quark-hadron chiral model

• Baryons interacting through mean fields + parity doubling + excluded volume
• Quarks in a PNJL-like approach
• Constrained to lattice data at $\mu_B = 0$ (high temperatures, low densities), empirical nuclear matter properties (low temperatures, high densities), neutron star properties, and gravitational-wave observations

Motornenko, V.V., Steinheimer, Schramm, Stoecker, arxiv:1905.00866 & A. Motornenko, talk this afternoon

A considerably more involved approach needed for a “complete” phase diagram
Status of the critical point at finite density
Critical point: Lattice perspective

• Estimating radius of convergence of Taylor expansion from leading coeffs.

  No hints for a critical point at $T > 135$ MeV
  “Small” $\mu_B/T < 2-3$ disfavored
  [see also A. Pasztor (Wuppertal-Budapest), 1807.09862]

  [HotQCD collaboration, 1701.04325]

• Analysis of the relativistic virial (cluster) expansion

  Expansion coefficients consistent with a
  Roberge-Weiss like ($\text{Im } [\mu_B/T] = \pi$)
  transition in the complex plane
  Critical point at $\mu_B/T < \pi$ disfavored

  [V.V., Steinheimer, Philipsen, Stoecker, 1711.01261]
Critical point: Heavy-ion perspective

Measurements of (high-order) fluctuations and correlations

STAR measurement of net-proton kurtosis shows non-monotonic energy dependence which might be associated with criticality.

Proper interpretation is challenging and requires **dynamical modeling** of critical fluctuations (critical mode) on top of hydro description + EoS with a critical point.

**Equations of state with a critical point:**
- 3D-Ising + Taylor [P. Parotto et al., 1805.05249], switching function [C. Plumberg et al., 1812.01684], Hagedorn bag-like model [V.V. et al., in preparation], etc.

**Implementing critical dynamics:**
- [Stephanov, Yin, 1712.10305; Nahrgang et al., 1804.05728; Akamatsu et al., 1811.05081]

All actively being developed
Summary

• Steady progress from lattice QCD on observables which constrain EoS at finite density. Reasonable (crossover) equation of state at moderate \( \mu_B \) can be obtained in effective models constrained to all available lattice data, including both the Taylor expansion coefficients and Fourier coefficients of the cluster expansion. 

*Examples: Cluster Expansion Model, Hagedorn bag-like model, Chiral mean-field model etc.*

• No critical point signals from lattice. “Small” \( \mu_B / T < 2-3 \) disfavored. Moderate collision energies are more promising in the search for the critical point.
Summary

• Steady progress from lattice QCD on observables which constrain EoS at finite density. Reasonable (crossover) equation of state at moderate $\mu_B$ can be obtained in effective models constrained to all available lattice data, including both the Taylor expansion coefficients and Fourier coefficients of the cluster expansion

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Thanks for your attention!
Backup slides
Radius of convergence from different models

Ideal HRG

Singularity in the nucleon Fermi-Dirac function

\[
\left[ \exp \left( \frac{\sqrt{m^2 + p^2} - \mu_B}{T} \right) + 1 \right]^{-1}
\]

EV-HRG & mean-field HRG

[Huovinen, Petreczky, 2017]

Repulsive baryonic interactions.

Singularity of the Lambert W function

van der Waals HRG

[V.V., Gorenstein, Stoecker, 2016]

Crossover singularities connected to the nuclear matter critical point at \( T \sim 20 \text{ MeV} \) and \( \mu_B \sim 900 \text{ MeV} \)

see also M. Stephanov, hep-lat/0603014

Cluster Expansion Model (CEM)

[V.V., Steinheimer, Philipsen, Stoecker, 2017]

Roberge-Weiss like transition: \( \text{Im} \frac{\mu_B}{T} = \pi \)

Taylor expansion likely divergent at \( \mu_B/T \geq 3-5 \), regardless of existence of the QCD critical point
Cluster expansion in fugacities

Expand in fugacity $\lambda_B = e^{\mu_B/T}$ instead of $\mu_B/T$ – a relativistic analogue of Mayer’s cluster expansion:

$$\frac{p(T, \mu_B)}{T^4} = \frac{1}{2} \sum_{k=-\infty}^{\infty} p_{|k|}(T) e^{k\mu_B/T} = \frac{p_0(T)}{2} + \sum_{k=1}^{\infty} p_k(T) \cosh(k\mu_B/T)$$

Net baryon density: $$\frac{\rho_B(T, \mu_B)}{T^3} = \sum_{k=1}^{\infty} b_k(T) \sinh(k\mu_B/T), \quad b_k \equiv kp_k$$

Analytic continuation to imaginary $\mu_B$ yields trigonometric Fourier series

$$\frac{\rho_B(T, i\tilde{\mu}_B)}{T^3} = i \sum_{k=1}^{\infty} b_k(T) \sin \left( \frac{k\tilde{\mu}_B}{T} \right)$$

with Fourier coefficients $$b_k(T) = \frac{2}{\pi} \frac{1}{T^4} \int_0^T d\tilde{\mu}_B \left[ \text{Im} \rho_B(T, i\tilde{\mu}_B) \right] \sin(k\tilde{\mu}_B/T)$$

Four leading coefficients $b_k$ computed in LQCD at the physical point

[V.V., A. Pasztor, Z. Fodor, S.D. Katz, H. Stoecker, 1708.02852]
Why cluster expansion is interesting?

Convergence properties of cluster expansion determined by singularities of thermodynamic potential in complex fugacity plane $\rightarrow$ encoded in the asymptotic behavior of the Fourier coefficients $b_k$

Examples:

- ideal quantum gas
  $$b_k \sim (\pm 1)^{k-1} \frac{e^{-km/T}}{k^{3/2}}$$
  Bose-Einstein condensation

- cluster expansion model
  $$b_k \sim (-1)^{k-1} \frac{\lambda_{br}^{-k}}{k}$$
  $|\lambda_{br}| = 1 \rightarrow$ Roberge-Weiss transition at imaginary $\mu_B$

- excluded volume model
  $$b_k \sim (-1)^{k-1} \frac{\lambda_{br}^{-k}}{k^{1/2}}$$
  No phase transition, but a singularity at a negative $\lambda$

- chiral crossover
  $$b_k \sim \frac{e^{-k\mu_c}}{k^{2-\alpha}} \sin(k\theta_c + \theta_0)$$
  Remnants of chiral criticality at $\mu_B = 0$

This work: signatures of a CP and a phase transition at finite density
HRG with repulsive baryonic interactions

Repulsive interactions with excluded volume (EV) \( V \rightarrow V - bN \)

[Hagedorn, Rafelski, '80; Dixit, Karsch, Satz, '81; Cleymans et al., '86; Rischke et al., Z. Phys. C '91]

HRG with baryonic EV:

\[
p_B(T, \mu_B) = p_B^{id}(T, \mu_B - b \rho_B)
\]

\[
b_k^{EV}(T) = (-1)^{k-1} \frac{2k^k}{k!} (bT^3)^{k-1} \left[ \frac{\phi_B(T)}{T^3} \right]^k
\]

V.V., A. Pasztor, Z. Fodor, S.D. Katz, H. Stoecker, 1708.02852

- Non-zero \( b_k(T) \) for \( k \geq 2 \) signal deviation from ideal HRG
- EV interactions between baryons \( (b \approx 1 \text{ fm}^3) \) reproduce lattice trend
Using estimators for radius of convergence

a) Ratio estimator:

\[ r_n = \left( \frac{(2n+2)(2n+1)\chi^B_{2n}}{\chi^B_{2n+2}} \right)^{1/2} \]

Ratio estimator is *unable* to determine the radius of convergence, nor to provide an upper or lower bound

b) Mercer-Roberts estimator:

\[ r_n = \left( \frac{c_{n+1}c_{n-1} - c_n^2}{c_n^2 c_{n+2} - c_n^2 c_{n+1}} \right)^{1/4} \]

\[ c_n = \frac{\chi^B_{2n}}{(2n)!} \]
CEM: Radius of convergence

Radius of convergence approaches Roberge-Weiss transition value

- At $T > T_{RW}$ expected $\left(\frac{\mu_B}{T}\right)_c = \pm i\pi$  [Roberge, Weiss, NPB '86] $T_{RW} \sim 208$ MeV
  
  [C. Bonati et al., 1602.01426]

- Complex plane singularities interfere with the search for CP
Expected asymptotics

- At low $T$/densities QCD $\simeq$ ideal hadron resonance gas

$$\frac{p^{\text{hrg}}(T, \mu_B)}{T^4} = \frac{\phi_M(T)}{T^3} + 2 \frac{\phi_B(T)}{T^3} \cosh \left( \frac{\mu_B}{T} \right),$$

$$\phi_B(T) = \sum_{i \in B} \int dm \rho_i(m) \frac{d_i m^2 T}{2\pi^2} K_2 \left( \frac{m}{T} \right),$$

$$p_0^{\text{hrg}}(T) = \frac{\phi_M(T)}{T^3}, \quad p_1^{\text{hrg}}(T) = \frac{2 \phi_B(T)}{T^3}, \quad p_k^{\text{hrg}}(T) \equiv 0, \ k \geq 2$$

- At high $T$ QCD $\simeq$ ideal gas of massless quarks and gluons

$$\frac{p^{\text{SB}}(T, \mu_B)}{T^4} = \frac{8\pi^2}{45} + \sum_{f=u,d,s} \left[ \frac{7\pi^2}{60} + \frac{1}{2} \left( \frac{\mu_B}{3T} \right)^2 + \frac{1}{4\pi^2} \left( \frac{\mu_B}{3T} \right)^4 \right],$$

$$p_0^{\text{SB}} = \frac{64\pi^2}{135}, \quad p_k^{\text{SB}} = \frac{(-1)^{k+1}}{k^2} \frac{4[3 + 4(\pi k)^2]}{27(\pi k)^2}, \quad b_k^{\text{SB}} = k p_k^{\text{SB}}.$$  

Lattice data explore intermediate, transition region $130 < T < 230$ MeV

*In this study we assume that $\mu_S = \mu_Q = 0$
Hagedorn resonance gas

HRG + exponential Hagedorn mass spectrum, e.g. as obtained from the bootstrap equation

\[ \rho(m) = A m^{-\alpha} \exp\left( \frac{m}{T_H} \right) \]

If Hagedorns are point-like, \( T_H \) is the limiting temperature
From limiting temperature to crossover

- A gas of extended objects $\rightarrow$ excluded volume
- Exponential spectrum of compressible QGP bags
- Both phases described by single partition function

[Gorenstein, Petrov, Zinovjev, PLB ’81; Gorenstein, W. Greiner, Yang, JPG ’98; I. Zakout et al., NPA ’07]

Crossover transition in bag-like model qualitatively compatible with LQCD

[Ferroni, Koch, PRC 79, 034905 (2009)]
Model formulation

Thermodynamic system of known hadrons and quark-gluon bags

**Mass-volume density** \( \rho(m, \nu; \lambda_B, \lambda_Q, \lambda_S) = \rho_H + \rho_Q \)

\[
\rho_H(m, \nu; \lambda_B, \lambda_Q, \lambda_S) = \sum_{i \in \text{HRG}} \lambda_B^b_i \lambda_Q^{q_i} \lambda_S^{\delta_i} d_i \delta(m - m_i) \delta(\nu - \nu_i) \quad \text{PDG hadrons}
\]

\[
\rho_Q(m, \nu; \lambda_B, \lambda_Q, \lambda_S) = C \nu^\gamma (m - B\nu)^\delta \exp \left\{ \frac{4}{3} [\sigma_Q \nu]^{1/4} (m - B\nu)^{3/4} \right\} \theta(\nu - V_0) \theta(m - B\nu) \quad \text{Quark-gluon bags}\]

Non-overlapping particles \( \rightarrow \) **isobaric (pressure) ensemble**

\[
\hat{Z}(T, s, \lambda_B, \lambda_Q, \lambda_S) = \int_0^\infty Z(T, \nu, \lambda_B, \lambda_Q, \lambda_S) e^{-s\nu} \, d\nu = [s - f(T, s, \lambda_B, \lambda_Q, \lambda_S)]^{-1}
\]

\[
f(T, s, \lambda_B, \lambda_Q, \lambda_S) = \int d\nu \int dm \rho(m, \nu; \lambda_B, \lambda_Q, \lambda_S) e^{-\nu s} \phi(T, m)
\]

The system pressure is \( p = Ts^* \) with \( s^* \) being the **rightmost** singularity of \( \hat{Z} \)
Mechanism for transition to QGP

The isobaric partition function, $\hat{Z}(T, s, \lambda) = [s - f(T, s, \lambda)]^{-1}$, has

- pole singularity $s_H = f(T, s_H, \lambda)$ **“hadronic” phase**

- singularity $s_B$ in the function $f(T, s, \lambda)$ due to the exponential spec' 

\[ \rho_B = T \, s_B = \frac{\sigma Q}{3} \, T^4 - B \]

MIT bag model EoS for QGP [Chodos+, PRD ’74; Baacke, APPB ’77]

1st order PT

“collision” of singularities

$s_H(T_c) = s_B(T_c)$

2nd order PT
crossover 

$s_H(T) > s_B(T)$ at all $T$
Crossover transition

Type of transition is determined by exponents $\gamma$ and $\delta$ of bag spectrum

Crossover seen in lattice, realized in model for $\gamma + \delta \geq -3$ and $\delta \geq -7/4$

Transcendental equation for pressure:

$$p(T, \lambda_B, \lambda_Q, \lambda_S) = T \sum_{i \in \text{HRG}} d_i \phi(T, m) \lambda_B^{b_i} \lambda_Q^{q_i} \lambda_S^{s_i} \exp \left( -\frac{m_i p}{4BT} \right)$$

$$+ \frac{C}{\pi} T^{5+4\delta} [\sigma_Q]^{\delta+1/2} [B + \sigma_Q T^4]^{3/2} \left( \frac{T}{p - p_B} \right)^{\gamma+\delta+3} \Gamma \left[ \gamma + \delta + 3, \frac{(p - p_B)V_0}{T} \right]$$

Solved numerically

Calculation setup:

$\gamma = 0, \ -3 \leq \delta \leq -\frac{1}{2}, \ B^{1/4} = 250 \ \text{MeV}, \ C = 0.03 \ \text{GeV}^{-\delta+2}, \ V_0 = 4 \ \text{fm}^3$

$$T_H = \left( \frac{3B}{\sigma_Q} \right)^{1/4} \approx 165 \ \text{MeV}$$
Thermodynamic functions

Pressure $p/T^4$

Energy density $\varepsilon/T^4$

- Crossover transition towards bag model EoS
- Dependence on $\delta$ is mild
- Approach to the Stefan-Boltzmann limit is too fast
- Peak in energy density, not seen on the lattice

Lattice data from 1309.5258 (Wuppertal-Budapest)
Nature of the transition

Filling fraction $= \frac{\langle V_{had} \rangle}{V}$

- Bags occupy almost whole space at large temperatures
- Strongest changes take place in the vicinity of $T_H$
- At $\delta < -7/4$ and $T \to \infty$ whole space — large bags with QGP
Conserved charges susceptibilities

Available from lattice QCD, not considered in this type of model before

\[
\chi_{mn}^{BSQ} = \frac{\partial^{l+m+n} p / T^4}{\partial (\mu_B / T)^l \partial (\mu_S / T)^m \partial (\mu_Q / T)^n}
\]
Bag model with massive quarks

Main source of quantitative disagreement comes from inaccuracy of the standard MIT bag model with massless quarks for describing QGP

Quasiparticle models suggest sizable thermal masses of quarks and gluons in high-temperature QGP [Peshier et al., PLB ’94; PRC ’00; PRC ’02]

Heavy-bag model: bag model EoS with non-interacting massive quarks and gluons and the bag constant [Ivanov et al., PRC 72, 025804 (2005)]

Massive quarks and gluons instead of massless ones:

$$
\sigma_Q(T, \lambda_B, \lambda_Q, \lambda_s) = \frac{8}{\pi^2 T^4} \int_0^\infty dk \frac{k^4}{\sqrt{k^2 + m_g^2}} \left[ \exp \left( \frac{\sqrt{k^2 + m_g^2}}{T} \right) - 1 \right]^{-1} \\
+ \sum_{f=u,d,s} \frac{3}{\pi^2 T^4} \int_0^\infty dk \frac{k^4}{\sqrt{k^2 + m_f^2}} \left[ \lambda_f^{-1} \exp \left( \frac{\sqrt{k^2 + m_f^2}}{T} \right) + 1 \right]^{-1} \\
+ \sum_{f=u,d,s} \frac{3}{\pi^2 T^4} \int_0^\infty dk \frac{k^4}{\sqrt{k^2 + m_f^2}} \left[ \lambda_f \exp \left( \frac{\sqrt{k^2 + m_f^2}}{T} \right) + 1 \right]^{-1}
$$
Bag model with massive quarks

Introduction of constituent masses leads to much better description of QGP

Parameters for the crossover model:

\[ m_u = m_d = 300 \text{ MeV}, \quad m_s = 350 \text{ MeV}, \quad m_g = 800 \text{ MeV}, \quad B^{1/4} = 200 \text{ MeV} \]

\[ \gamma = 0, \quad \delta = -2, \quad C = 0.03, \quad V_0 = 8 \text{ fm}^3 \]

\[ T_H \approx 167 \text{ MeV} \]
Hagedorn model: Thermodynamic functions

- Semi-quantitative description of lattice data
- Peak in energy density gone!
Hagedorn model: Thermodynamic functions

Trace anomaly \( (\varepsilon - 3p)/T^4 \)

Speed of sound \( c_s^2 = dp/d\varepsilon \)
Hagedorn model: Susceptibilities

\[ \chi_{lmn}^{BSQ} = \frac{\partial^{l+m+n} \rho / T^4}{\partial (\mu_B / T)^l \partial (\mu_S / T)^m \partial (\mu_Q / T)^n} \]

Lattice data from 1112.4416 (Wupperatal-Budapest), 1203.0784 (HotQCD)
Hagedorn model: Baryon-strangeness ratio

\[ C_{BS} = -\frac{3\chi_{11}^{BS}}{\chi_S^2} \]

Useful diagnostic of QCD matter

[V. Koch, Majumder, Randrup, PRL 95, 182301 (2005)]

Well consistent with lattice QCD
Hagedorn model: Higher-order susceptibilities

Higher-order susceptibilities are particularly sensitive probes of the parton-hadron transition and possible remnants of criticality at $\mu_B = 0$

\[ \frac{\chi_4^B}{\chi_2^B} \] net baryon

\[ \frac{\chi_4^S}{\chi_2^S} \] net strangeness

- Drop of $\frac{\chi_4^B}{\chi_2^B}$ caused by repulsive interactions which ensure crossover transition to QGP
- Peak in $\frac{\chi_4^S}{\chi_2^S}$ is an interplay of the presence of multi-strange hyperons and repulsive interactions

Lattice data from 1305.6297 & 1805.04445 (Wuppertal-Budapest), 1708.04897 (HotQCD)