Towards the QCD equation of state at finite density

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QCD phase diagram: towards finite density

- QCD EoS at $\mu_B = 0$ available from lattice QCD
- QCD EoS at finite density necessary for many applications, including hydro modeling of heavy-ion collisions at RHIC, SPS, FAIR energies
- Implementation of the QCD critical point necessary to look for its signatures
Outline

1. Taylor expansion from lattice QCD
   - Model-independent method with a limited scope (small $\mu_B/T$)
   - State-of-the-art and estimates for radius of convergence

2. Lattice-based effective models
   - Cluster expansion model (CEM)
   - Hagedorn bag-like model

3. Signatures of the critical point at finite density
   - Exponential suppression of Fourier coefficients
   - Extracting the location of singularities from the lattice data
Finite $\mu_B$ EoS from Taylor expansion

$$\frac{p(T, \mu_B)}{T^4} = \frac{p(T, 0)}{T^4} + \frac{\chi^B_2(T, 0)}{2!} (\frac{\mu_B}{T})^2 + \frac{\chi^B_4(T, 0)}{4!} (\frac{\mu_B}{T})^4 + \ldots$$

$\chi^B_{k}$ – cumulants of net baryon distribution, computed up to $\chi^B_8$

- Off-diagonal susceptibilities also available → incorporate conservation laws $n_S = 0$, $n_Q/n_B = 0.4$
- Method inherently limited to “small” $\mu_B/T$, within convergence radius

[HotQCD collaboration, 1701.04325] [Wuppertal-Budapest collaboration, 1805.04445]
A truncated Taylor expansion only useful within the *radius of convergence*. Its value is a priori unknown. Any singularity in *complex* $\mu_B$ plane will limit the convergence, it does not have to be a phase transition or a critical point.
Taylor expansion and radius of convergence

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**An example:** HRG model with a baryonic excluded volume (EV) $V \rightarrow V - bN$

$$p(T, \mu_B) \sim W \left[ b \phi_B(T) e^{\mu_B/T} \right],$$

$b \simeq 1 \text{ fm}^3$

*Constrained to LQCD data [V.V. et al., 1708.02852]*

Lambert $W(z)$ function has a branch cut singularity at $z = -e^{-1}$, corresponds to a negative fugacity

[Taradiy, Motornenko, V.V., Gorenstein, Stoecker, 1904.08259]
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Radius of convergence from different models

Ideal HRG

Singularity in the nucleon Fermi-Dirac function

$$\left[ \exp \left( \frac{\sqrt{m^2 + p^2} - \mu_B}{T} \right) + 1 \right]^{-1}$$

EV-HRG & mean-field HRG

[\text{V.V.}+, 1708.02852] \hspace{1cm} [\text{Huovinen, Petreczky, 1708.02852}]

Repulsive baryonic interactions.

Singularity of the Lambert W function

van der Waals HRG

[\text{V.V.}, Gorenstein, Stoecker, 1609.03975]

Crossover singularities connected to the nuclear matter critical point at $T \sim 20$ MeV and $\mu_B \sim 900$ MeV

see also M. Stephanov, hep-lat/0603014

Cluster Expansion Model (CEM)

[\text{V.V.}, Steinheimer, Philipsen, Stoecker, 1711.01261]

Roberge-Weiss like transition: $\text{Im} \frac{\mu_B}{T} = \pi$

Taylor expansion likely divergent at $\mu_B/T \geq 3-5$, regardless of existence of the QCD critical point
Recent Taylor-based EoS parameterizations

\[
\frac{p}{T^4} = \sum_{i,j,k} \chi_{i,j,k}^{BQS}(T) \left( \frac{\mu_B}{T} \right)^i \left( \frac{\mu_Q}{T} \right)^j \left( \frac{\mu_S}{T} \right)^k
\]

\[
\frac{p}{T^4} = \sum_{i \in \text{hrg}} T \phi_i^{id}(T) e^{b_i \mu_B/T} e^{q_i \mu_Q/T} e^{s_i \mu_S/T}
\]

(a) NEOS BQS

- Includes the three conserved charges and conservation laws, no criticality
- Probably best one can do with Taylor expansion. Applications: RHIC BES

[Monnai, Schenke, Shen, 1902.05095]  
[Noronha-Hostler, Parotto, Ratti, Stafford, 1902.06723]
Truncated Taylor expansion and imaginary $\mu_B$

Are we using all information available from lattice? Consider relativistic virial expansion (Laurent series in fugacity) and imaginary $\mu_B$

$$\left. \frac{\rho_B}{T^3} \right|_{\mu_B=i\theta_B T} = i \sum_{k=1}^{\infty} b_k(T) \sin(k\theta_B) \quad \Rightarrow \quad b_k(T) = -\frac{2i}{\pi} \int_{0}^{\pi} \frac{\rho_B(T, i\theta_B T)}{T^3} \sin(k\theta_B) \, d\theta_B$$

Relativistic virial/cluster expansion

Fourier coefficients
Truncated Taylor expansion and imaginary $\mu_B$

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Relativistic virial/cluster expansion

Fourier coefficients

**Lines:** Taylor expansion up to $\chi_B^4$ using lattice data, as in 1902.06723

**Symbols:** Lattice data for $b_k$ from imaginary $\mu_B$

[V.V., Pasztor, Fodor, Katz, Stoecker, 1708.02852]

Quite some room for improvement at $T<200$ MeV
Cluster Expansion Model — CEM

a model for QCD equation of state at finite baryon density constrained to both susceptibilities and Fourier coefficients

Cluster Expansion Model (CEM)

Model formulation:

• Cluster expansion for baryon number density

\[ \frac{\rho_B(T, \mu_B)}{T^3} = \chi_1^B(T, \mu_B) = \sum_{k=1}^{\infty} b_k(T) \sinh(k\mu_B/T) \]

• \( b_1(T) \) and \( b_2(T) \) are model input from lattice QCD

• All higher order coefficients are predicted: \( b_k(T) = \alpha_k^{SB} \frac{[b_2(T)]^{k-1}}{[b_1(T)]^{k-2}} \)

Physical picture: Hadron gas with repulsion at moderate \( T \), QGP-like phase at high \( T \)

Summed analytic form:

\[ \frac{\rho_B(T, \mu_B)}{T^3} = -\frac{2}{27\pi^2} \frac{\hat{b}_2^2}{\hat{b}_2} \left\{ 4\pi^2 \left[ \text{Li}_1(x_+) - \text{Li}_1(x_-) \right] + 3 \left[ \text{Li}_3(x_+) - \text{Li}_3(x_-) \right] \right\} \]

\[ \hat{b}_k = \frac{b_k(T)}{b_k^{SB}}, \quad x_\pm = -\frac{\hat{b}_2}{\hat{b}_1} e^{\pm\mu_B/T}, \quad \text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \]

Regular behavior at real \( \mu_B \) \( \rightarrow \) no-critical-point scenario
CEM: Fourier coefficients

\[ b_k(T) \]

\[ \alpha_k(T) \equiv b_k(T) \frac{[b_1(T)]^{k-2}}{[b_2(T)]^{k-1}} \]

CEM: \( b_1(T) \) and \( b_2(T) \) as input → consistent description of \( b_3(T) \) and \( b_4(T) \)

Lattice data on \( b_{3,4}(T) \) inconclusive at \( T \leq 170 \) MeV
CEM: Baryon number susceptibilities

\[ \chi_k^B(T, \mu_B) = -\frac{2}{27\pi^2} \frac{\hat{b}_1^2}{b_2} \left\{ 4\pi^2 \left[ \text{Li}_{2-k}(x_+) + (-1)^k \text{Li}_{2-k}(x_-) \right] + 3 \left[ \text{Li}_{4-k}(x_+) + (-1)^k \text{Li}_{4-k}(x_-) \right] \right\} \]

Lattice data from 1805.04445 (Wuppertal-Budapest), 1701.04325 & 1708.04897 (HotQCD)
CEM: Equation of state

\[
\frac{p(T, \mu_B)}{T^4} = \frac{p_0(T)}{2} - \frac{2}{27\pi^2} \frac{\hat{b}_1^2}{\hat{b}_2} \left\{ 4\pi^2 \left[ \text{Li}_2(x_+) - \text{Li}_2(x_-) \right] + 3 \left[ \text{Li}_4(x_+) - \text{Li}_4(x_-) \right] \right\}
\]

**Input:** \( p_0(T), b_{1,2}(T) \) ← parametrized LQCD + HRG

Tabulated CEM EoS available at [https://fias.uni-frankfurt.de/~vovchenko/cem_table/](https://fias.uni-frankfurt.de/~vovchenko/cem_table/)

Currently restricted to single chemical potential (\( \mu_B \)) and no critical point
Hagedorn (bag-like) resonance gas model with repulsive interactions
exactly solvable model with a (phase) transition between hadronic matter and QGP

[Gorenstein, Petrov, Zinovjev, PLB '81; Gorenstein, W. Greiner, Yang, JPG '98; Ferroni, Koch, PRC '09]

Here the model equation of state is constrained to lattice QCD

Hagedorn bag-like model: formulation

- HRG + quark-gluon bags
  \[ \rho_Q(m, \nu) = C \nu^\gamma (m - B\nu)^\delta \exp \left\{ \frac{4}{3} \left[ \sigma_Q \right]^{1/4} \nu^{1/4} (m - B\nu)^{3/4} \right\} \]
- Non-overlapping particles (excluded volume correction)
  \[ V \rightarrow V - bN \]
- Isobaric (pressure) ensemble \((T, V, \mu) \rightarrow (T, s, \mu)\)
- Massive (thermal) partons (new element)
Hagedorn bag-like model: formulation

- HRG + quark-gluon bags \( \rho_Q(m, \nu) = C \nu^\gamma (m - B \nu)^\delta \exp \left\{ \frac{4}{3} \sigma_Q \frac{\nu^{1/4}}{\sqrt{\pi}} (m - B \nu)^{3/4} \right\} \)
- Non-overlapping particles (excluded volume correction)
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- Massive (thermal) partons (new element)

Resulting picture: transition (crossover, 1\textsuperscript{st} order, 2\textsuperscript{nd} order, etc.) between HRG and MIT bag model EoS, within single partition function

"Crossover" parameter set

\( \gamma = 0, \quad \delta = -2, \quad C = 0.03, \quad V_0 = 8 \text{ fm}^3 \)
\( m_u = m_d = 300 \text{ MeV}, \quad m_s = 350 \text{ MeV} \)
\( m_g = 800 \text{ MeV}, \quad B^{1/4} = 200 \text{ MeV} \)
Hagedorn model: Susceptibilities

Lattice data from 1112.4416 (Wuppertal-Budapest), 1203.0784 (HotQCD)
Hagedorn model: Susceptibilities and Fourier

\[ \chi_4^B / \chi_2^B \]

\[ \chi_6^B / \chi_2^B \]

Massive quarks
\[ B^{1/4} = 200 \text{ MeV} \]
\[ \gamma = 0, \delta = -2 \]
\[ V_0 = 8 \text{ fm}^3 \]

\[ \chi_8^B \]

\[ b_k \]
Hagedorn model: Finite baryon density

- Crossover transition to a QGP-like phase in both the $T$ and $\mu_B$ directions
- Essentially a built-in “switching” function between HRG and QGP, thermodynamically consistent by construction (single partition function)
- Critical point/phase transition at finite $\mu_B$ can be incorporated through $\mu_B$-dependence of $\gamma$ and $\delta$ exponents in bag spectrum

see Gorenstein, Gazdzicki, Greiner, Phys. Rev. C (2005)

More on this at SQM2019
Signatures of a critical point/phase transition at finite baryon density

currently no indications for the location of QCD critical point from lattice data, “small” $\mu_B/T \leq 2-3$ disfavored

[Bazavov et al., 1701.04325; V.V. et al. 1711.01261; Fodor et al., 1807.09862]

Recent works incorporating a CP to study its signatures in heavy-ion collisions:

- P. Parotto et al., arxiv:1805.05249 – 3D Ising model, matched with LQCD susceptibilities, CP location can be varied

- C. Plumberg, T. Welle, J. Kapusta, arxiv:1812.01684 – CP through a switching function, location can be varied

- R. Critelli et al., arxiv:1706.00455 – holographic gauge/gravity corr., CP at “small” energies

This work: signatures of a critical point and a phase transition at finite density in the cluster expansion (imaginary $\mu_B$ LQCD observables)
A model with a phase transition

Our starting point is a single-component fluid. We are looking for a theory with a phase transition where Mayer’s cluster expansion

$$\frac{n(T, \lambda)}{T^3} = \frac{1}{2} \sum_{k=1}^{\infty} b_k(T) \lambda^k$$

can be worked out explicitly. The “tri-virial” model (TVM)

$$p(T, n) = T n + T \left( b - \frac{a}{T} \right) n^2 + T b^2 n^3$$

which is the vdW equation truncated at $n^3$, has the required features.

Critical point:

$$\left( \frac{\partial p}{\partial n} \right)_T = 0, \quad \left( \frac{\partial^2 p}{\partial n^2} \right)_T = 0$$

$$T_c = \frac{\sqrt{3} - 1}{2} \frac{a}{b}, \quad n_c = \frac{1}{\sqrt{3} b}, \quad p_c = \frac{3 - \sqrt{3}}{18} \frac{a}{b^2}$$

isotherms
TVM in the grand canonical ensemble (GCE)

Transformation from \((T, n)\) variables to \((T, \mu)\) [or \((T, \lambda)\)] variables

\[
p(T, n) = T n + T \left( b - \frac{a}{T} \right) n^2 + T b^2 n^3
\]

\[
p(T, n) = - \left( \frac{\partial F}{\partial V} \right)_{T,N} \quad \Rightarrow \quad F(T, V, N) \quad \Rightarrow \quad \mu = \left( \frac{\partial F}{\partial N} \right)_{T,V}
\]

\[
\lambda = \frac{n}{\phi(T)} \exp \left[ \frac{3}{2} (bn)^2 + 2n \left( b - \frac{a}{T} \right) \right], \quad \lambda \equiv e^{\mu/T}
\]

The defining transcendental equation for the GCE particle number density \(n(T, \lambda)\)

This equation encodes the analytic properties of the grand potential associated with a phase transition.
TVM: the branch points

\[ \lambda = \frac{n}{\phi(T)} \exp \left[ \frac{3}{2} (bn)^2 + 2n \left( b - \frac{a}{T} \right) \right] \]

The defining equation permits multiple solutions therefore \( n(T, \lambda) \) is multi-valued and has singularities – the branch points:

\[ \left( \frac{\partial \lambda}{\partial n} \right)_T = 0 \quad \Rightarrow \quad 3(bn_{br})^2 + 2 \left( 1 - \frac{a}{bT} \right) bn_{br} + 1 = 0 \]
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Solutions:

- \( T > T_C \): two c.c. roots \( n_{br1} = (n_{br2})^* \)

- \( T = T_C \): \( n_{br1} = n_{br2} = n_c \)

- \( T < T_C \): two real roots \( n_{sp1} \) and \( n_{sp2} \)

see also M. Stephanov, Phys. Rev. D 73, 094508 (2006)
TVM: Mayer’s cluster expansion

$$\lambda = \frac{n}{\phi(T)} \exp \left[ \frac{3}{2} (bn)^2 + 2n \left( b - \frac{a}{T} \right) \right]$$

$$\frac{n(T, \lambda)}{T^3} = \frac{1}{2} \sum_{k=1}^{\infty} b_k(T) \lambda^k$$
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\[ \frac{n(T, \lambda)}{T^3} = \frac{1}{2} \sum_{k=1}^{\infty} b_k(T) \lambda^k \]

Lagrange inversion theorem

If \( y = f(x) \), \( y_0 = f(x_0) \), \( f'(x_0) \neq 0 \), then

\[ 3.6.6 \]

\[ x = x_0 + \sum_{k=1}^{\infty} \frac{(y-y_0)^k}{k!} \left[ \frac{d^{k-1}}{dx^{k-1}} \left\{ \frac{x-x_0}{f(x)-y_0} \right\}^k \right]_{x=x_0} \]

from Abramowitz, Stegun, “Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables”
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\[ y \equiv \lambda, \quad x \equiv n, \quad f(x) \equiv \lambda(n; T) \]

\[ \lambda_0 = 0, \quad n_0 = 0 \]

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from Abramowitz, Stegun, “Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables”

\[ \frac{n(T, \lambda)}{T^3} = \frac{1}{2} \sum_{k=1}^{\infty} b_k(T) \lambda^k \]

Result:

\[ b_k(T) = 2 \frac{\phi(T)}{T^3} \left[ b \phi(T) \right]^{k-1} \frac{1}{k!} \left( \frac{3k}{2} \right)^{k-1} \lim_{x \to 0} \frac{d^{k-1}}{dx^{k-1}} \exp \left[ -2 \sqrt{\frac{2k}{3}} \left( 1 - \frac{a}{bT} \right) x - x^2 \right] \]
TVM: Mayer’s cluster expansion

\[ b_k(T) = 2 \frac{\phi(T)}{T^3} \left[ b \phi(T) \right]^{k-1} \frac{1}{k!} \left( \frac{3k}{2} \right)^{\frac{k-1}{2}} \lim_{x \to 0} \frac{d^{k-1}}{dx^{k-1}} \exp \left[ -2 \sqrt{\frac{2k}{3}} \left( 1 - \frac{a}{bT} \right) x - x^2 \right] \]
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Generating function of Hermite polynomials:

\[ e^{2tx - \frac{1}{2}x^2} = \sum_{n=0}^{\infty} H_n(t) \frac{x^n}{n!} \]
TVM: Mayer’s cluster expansion

\[ b_k(T) = 2 \frac{\phi(T)}{T^3} \left[ b \phi(T) \right]^{k-1} \frac{1}{k!} \left( \frac{3k}{2} \right)^{\frac{k-1}{2}} \lim_{x \to 0} \frac{d^{k-1}}{dx^{k-1}} \exp \left[ -2 \sqrt{\frac{2k}{3}} \left( 1 - \frac{a}{bT} \right) x - x^2 \right] \]

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A potentially non-trivial behavior of cluster integrals \( b_k \) associated with a presence of a phase transition is determined by the Hermite polynomials
Asymptotic behavior of \( b_k \) determined mainly by Hermite polynomials

\[
b_k \sim H_{k-1} \left[ -\sqrt{\frac{2k}{3}} \left( 1 - \frac{a}{b \, T} \right) \right]
\]

**A caveat:** both the argument and the index of \( H \) tend to large values.
Asymptotic behavior of cluster integrals

Asymptotic behavior of $b_k$ determined mainly by Hermite polynomials

$$b_k \sim H_{k-1} \left[ -\sqrt{\frac{2k}{3}} \left( 1 - \frac{a}{bT} \right) \right]$$

**A caveat:** both the argument and the index of $H$ tend to large values. Such a case was analyzed in [D. Dominici, arXiv:math/0601078]

1) $x > \sqrt{2n}$

\[
T < T_C \quad \Rightarrow \quad H_n(x) \xrightarrow{n \to \infty} \exp \left[ \frac{x^2 - \sigma x - n}{2} + n \ln(\sigma + x) \right] \sqrt{\frac{1}{2}} \left( 1 + \frac{x}{\sigma} \right), \quad \sigma = \sqrt{x^2 - 2n}
\]

2) $x \approx \sqrt{2n}$

\[
T = T_C \quad \Rightarrow \quad H_n(x) \xrightarrow{n \to \infty} \exp \left[ \frac{n}{2} \ln(2n) - \frac{3}{2} n + \sqrt{2n} x \right] \sqrt{2\pi n^{1/6}} \text{Ai} \left[ \sqrt{2} (x - \sqrt{2n}) n^{1/6} \right]
\]

3) $|x| < \sqrt{2n}$

\[
T > T_C \quad \Rightarrow \quad H_n \left[ \sqrt{2n} \sin \theta \right] \xrightarrow{n \to \infty} \sqrt{\frac{2}{\cos \theta}} \exp \left\{ \frac{n}{2} \left[ \ln(2n) - \cos(2\theta) \right] \right\} \cos \left\{ n \left[ \frac{1}{2} \sin(2\theta) + \theta - \frac{\pi}{2} \right] + \frac{\theta}{2} \right\}
\]

Asymptotic behavior changes as one traverses the critical temperature
Asymptotic behavior of cluster integrals

1) \( T < T_c \) : 
   \[ b_k(T) \overset{k \to \infty}{\sim} A_- \frac{e^{-k \mu_{sp1}}}{k^{3/2}} \]

   \( b_k \) see the spinodal point of a first-order phase transition

2) \( T = T_c \) : 
   \[ b_k(T) \overset{k \to \infty}{\sim} A_c \frac{e^{-k \mu_c}}{k^{4/3}} \]

   \( b_k \) see the critical point

3) \( T > T_c \) : 
   \[ b_k(T) \overset{k \to \infty}{\sim} A_+ \frac{e^{-k \mu_{br}}}{k^{3/2}} \sin \left( k \frac{\mu_{br}^l}{T} + \frac{\theta_0}{2} \right) \]

   crossover singularities \( \rightarrow \) oscillatory behavior of \( b_k \)

Behavior expected to be universal for the mean-field universality class, the likely effect of a change in universality class (e.g. 3D-Ising) is a modification of the power-law exponents
Applications to the QCD thermodynamics

TVM for “baryonic” pressure: \[ p_B(T, \mu) = T n_B + T \left( b - \frac{a}{T} \right) n_B^2 + T b^2 n_B^3 \]

**Symmetrization:** \( \mu_B \rightarrow -\mu_B \)

\[ p = p_B(T, \mu_B) + p_B(T, -\mu_B) + p_M(T) \]

“baryons” “anti-baryons” “mesons”

\[ \frac{\rho_B(T, i\theta_B T)}{T^3} = i \sum_{k=1}^{\infty} b_k(T) \sin(k \theta_B T) \]

Cluster integrals become Fourier coefficients (as long as \( b_k(T) \) \( k \rightarrow \infty \) 0 holds)

**Expected asymptotics**

\[ b_k(T) \xrightarrow{k \rightarrow \infty} A e^{-k \mu_{br}^R / T} k^\alpha \sin \left( k \frac{\mu_{br}^I}{T} + \frac{\theta_0}{2} \right), \quad \frac{\mu_{br}^R}{T} = \text{Re} \left[ \frac{\mu_{br}}{T} \right], \quad \frac{\mu_{br}^I}{T} = \text{Im} \left[ \frac{\mu_{br}}{T} \right] \]

*Can be tested in lattice QCD at imaginary chemical potential*
Extracting information from Fourier coefficients

\[ b_k(T) \xrightarrow{k \to \infty} A \frac{e^{-k \frac{\mu_{br}^R}{T}}}{k^\alpha} \sin \left( k \frac{\mu_{br}}{T} + \frac{\theta_0}{2} \right) \]

Real part of the limiting singularity determines the exponential suppression of Fourier coefficients.

To extract \( \text{Re}[\mu_{br}/T] \) fit \( b_k \) with \( \log |b_k| = A - (3/2) \log k - k \text{ Re} \left[ \frac{\mu_{br}}{T} \right] \)
Extracting information from Fourier coefficients

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\[
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\]

Illustration: TVM parameters fixed to a CP at \( T_c = 120 \text{ MeV}, \mu_c = 527 \text{ MeV} \)

<table>
<thead>
<tr>
<th>( T ) [MeV]</th>
<th>Fit to ( b_1-b_4 )</th>
<th>True value</th>
</tr>
</thead>
<tbody>
<tr>
<td>150</td>
<td>2.31</td>
<td>2.50</td>
</tr>
<tr>
<td>120</td>
<td>4.24</td>
<td>4.39</td>
</tr>
<tr>
<td>100</td>
<td>6.11</td>
<td>6.18</td>
</tr>
</tbody>
</table>
Fourier coefficients from lattice

Lattice QCD data (Wuppertal-Budapest), physical quark masses

[V.V., Pasztor, Fodor, Katz, Stoecker, PLB 775, 71 (2017)]
Fourier coefficients from lattice

Lattice QCD data (Wuppertal-Budapest), physical quark masses

[V.V., Pasztor, Fodor, Katz, Stoecker, PLB 775, 71 (2017)]

Can one extract useful information from lattice data?
Extracting singularities from lattice data

Fit lattice data with an **ansatz**:

\[ \log |b_k| = A - \alpha \log k - k \text{ Re} \left[ \frac{\mu_{\text{br}}}{T} \right] \]
Extracting singularities from lattice data

Fit lattice data with an ansatz:

$$\log |b_k| = A - \alpha \log k - k \text{Re} \left[ \frac{\mu_{br}}{T} \right]$$

Quite similar results for $1 \leq \alpha \leq 2$
Extracting singularities from lattice data

Fit lattice data with an ansatz:

\[
\log |b_k| = A - \alpha \log k - k \Re \left[ \frac{\mu_{br}}{T} \right]
\]

- \(b_k \sim (-1)^{k-1}\) in the data

Quite similar results for \(1 \leq \alpha \leq 2\)

\[\rightarrow \quad \Im \left[ \frac{\mu_{br}}{T} \right] \lesssim \pi\]
Extracting singularities from lattice data

Fit lattice data with an ansatz:

\[ \log |b_k| = A - \alpha \log k - k \text{Re} \left[ \frac{\mu_{br}}{T} \right] \]

- \( b_k \sim (-1)^{k-1} \) in the data

- \( \text{Re} \left[ \frac{\mu_{br}}{T} \right] \approx 0 \) for \( T \geq 200 \text{ MeV} \)

\( \Rightarrow \) singularity at purely imaginary \( \mu_B \)

\( \Rightarrow \) Roberge-Weiss transition?

LQCD: \( T_{RW} \approx 208 \text{ MeV} \) [C. Bonati et al., 1602.01426]
Summary

• Steady progress from lattice QCD on observables which constrain EoS at finite density. Reasonable (crossover) equation of state at moderate $\mu_B$ can be obtained in effective models constrained to all available lattice data, including both the Taylor expansion coefficients and Fourier coefficients of the cluster expansion

Examples: Cluster Expansion Model, Hagedorn bag-like model, etc.

• Location of thermodynamic singularities, e.g. the QCD critical point, can be extracted from LQCD at imaginary chemical potential via exponential suppression of Fourier coefficients.
Summary

• Steady progress from lattice QCD on observables which constrain EoS at finite density. Reasonable (crossover) equation of state at moderate $\mu_B$ can be obtained in effective models constrained to all available lattice data, including both the Taylor expansion coefficients and Fourier coefficients of the cluster expansion.

  Examples: Cluster Expansion Model, Hagedorn bag-like model, etc.

• Location of thermodynamic singularities, e.g. the QCD critical point, can be extracted from LQCD at imaginary chemical potential via exponential suppression of Fourier coefficients.

Thanks for your attention!
Backup slides
Cluster expansion in fugacities

Expand in fugacity $\lambda_B = e^{\mu_B/T}$ instead of $\mu_B/T$ – a relativistic analogue of Mayer’s cluster expansion:

$$\frac{p(T, \mu_B)}{T^4} = \frac{1}{2} \sum_{k=-\infty}^{\infty} p_k(T) e^{k\mu_B/T} = \frac{p_0(T)}{2} + \sum_{k=1}^{\infty} p_k(T) \cosh(k\mu_B/T)$$

Net baryon density: $\frac{\rho_B(T, \mu_B)}{T^3} = \sum_{k=1}^{\infty} b_k(T) \sinh(k\mu_B/T), \quad b_k \equiv kp_k$

Analytic continuation to imaginary $\mu_B$ yields trigonometric Fourier series

$$\frac{\rho_B(T, i\tilde{\mu}_B)}{T^3} = i \sum_{k=1}^{\infty} b_k(T) \sin\left(\frac{k\tilde{\mu}_B}{T}\right)$$

with Fourier coefficients $b_k(T) = \frac{2}{\pi T^4} \int_0^{\pi T} d\tilde{\mu}_B \left[\text{Im} \rho_B(T, i\tilde{\mu}_B)\right] \sin(k\tilde{\mu}_B/T)$

Four leading coefficients $b_k$ computed in LQCD at the physical point

[V.V., A. Pasztor, Z. Fodor, S.D. Katz, H. Stoecker, 1708.02852]
Why cluster expansion is interesting?

Convergence properties of cluster expansion determined by singularities of thermodynamic potential in complex fugacity plane \( \to \) encoded in the asymptotic behavior of the Fourier coefficients \( b_k \).

**Examples:**

- **ideal quantum gas** \( b_k \sim (\pm 1)^{k-1} \frac{e^{-km/T}}{k^{3/2}} \)
  
  Bose-Einstein condensation

- **cluster expansion model** \( b_k \sim (-1)^{k-1} \frac{|\lambda_{br}|^{-k}}{k} \)
  
  \(|\lambda_{br}| = 1 \rightarrow Roberge-Weiss transition at imaginary \( \mu_B \))

- **excluded volume model** \( b_k \sim (-1)^{k-1} \frac{|\lambda_{br}|^{-k}}{k^{1/2}} \)
  
  No phase transition, but a singularity at a negative \( \lambda \)

- **chiral crossover** \( b_k \sim \frac{e^{-k\tilde{\mu}_c}}{k^{2-\alpha}} \sin(k\theta_c + \theta_0) \)
  
  Remnants of chiral criticality at \( \mu_B = 0 \)

**This work:** signatures of a CP and a phase transition at finite density
HRG with repulsive baryonic interactions

Repulsive interactions with excluded volume (EV) $V \rightarrow V - bN$

[Hagedorn, Rafelski, '80; Dixit, Karsch, Satz, '81; Cleymans et al., '86; Rischke et al., Z. Phys. C '91]

HRG with baryonic EV:

$$p_B(T, \mu_B) = p_B^{id}(T, \mu_B - b \mu_B)$$

$$b_k^{EV}(T) = (-1)^{k-1} \frac{2k^k}{k!} (b \frac{T^3}{T^3})^{k-1} \left[ \frac{\phi_B(T)}{T^3} \right]^k$$

- Non-zero $b_k(T)$ for $k \geq 2$ signal deviation from ideal HRG
- EV interactions between baryons ($b \approx 1 \text{ fm}^3$) reproduce lattice trend
Using estimators for radius of convergence

a) Ratio estimator:

\[ r_n = \left| \frac{(2n+2)(2n+1)\chi^B_{2n}}{\chi^B_{2n+2}} \right|^{1/2} \]

Ratio estimator is *unable* to determine the radius of convergence, nor to provide an upper or lower bound.

b) Mercer-Roberts estimator:

\[ r_n = \left| \frac{c_{n+1} c_{n-1} - c_n^2}{c_{n+2} c_n - c_{n+1}^2} \right|^{1/4} \]

\[ c_n = \frac{\chi^B_{2n}}{(2n)!} \]
CEM: Radius of convergence

Radius of convergence approaches Roberge-Weiss transition value

- At $T > T_{RW}$ expected $\left[ \frac{\mu_B}{T} \right]_c = \pm i\pi$ \cite{Roberge, Weiss, NPB ’86} $T_{RW} \sim 208$ MeV \cite{C. Bonati et al., 1602.01426}
- Complex plane singularities interfere with the search for CP
Expected asymptotics

- At low $T$/densities QCD $\simeq$ ideal hadron resonance gas

$$\frac{p_{hrg}(T, \mu_B)}{T^4} = \frac{\phi_M(T)}{T^3} + 2 \frac{\phi_B(T)}{T^3} \cosh \left( \frac{\mu_B}{T} \right),$$

$$\phi_B(T) = \sum_{i \in B} \int dm \rho_i(m) \frac{d_i m^2 T}{2\pi^2} K_2 \left( \frac{m}{T} \right),$$

$$p_{hrg}^0(T) = \frac{\phi_M(T)}{T^3}, \quad p_{hrg}^1(T) = \frac{2 \phi_B(T)}{T^3}, \quad p_{hrg}^k(T) \equiv 0, \quad k \geq 2$$

- At high $T$ QCD $\simeq$ ideal gas of massless quarks and gluons

$$\frac{p_{SB}(T, \mu_B)}{T^4} = \frac{8\pi^2}{45} + \sum_{f=u,d,s} \left[ \frac{7\pi^2}{60} + \frac{1}{2} \left( \frac{\mu_B}{3T} \right)^2 + \frac{1}{4\pi^2} \left( \frac{\mu_B}{3T} \right)^4 \right],$$

$$p_{SB}^0 = \frac{64\pi^2}{135}, \quad p_{SB}^k = \frac{(-1)^{k+1}}{k^2} \frac{4 [3 + 4 (\pi k)^2]}{27 (\pi k)^2}, \quad b_{SB}^k = k p_{SB}^k.$$

Lattice data explore intermediate, transition region $130 < T < 230$ MeV

*In this study we assume that $\mu_S = \mu_Q = 0$
Hagedorn resonance gas

HRG + exponential Hagedorn mass spectrum, e.g. as obtained from the bootstrap equation [Hagedorn ‘65; Frautschi, ‘71]

If Hagedorns are point-like, $T_H$ is the limiting temperature

[Beitel, Gallmeister, Greiner, 1402.1458]
From limiting temperature to crossover

- A gas of extended objects $\rightarrow$ excluded volume
- Exponential spectrum of compressible QGP bags
- Both phases described by single partition function

[Ferroni, Koch, PRC 79, 034905 (2009)]

Crossover transition in bag-like model qualitatively compatible with LQCD
Model formulation

Thermodynamic system of known hadrons and quark-gluon bags

**Mass-volume density** \( \rho(m, v; \lambda_B, \lambda_Q, \lambda_S) = \rho_H + \rho_Q \)

\[
\rho_H(m, v; \lambda_B, \lambda_Q, \lambda_S) = \sum_{i \in \text{HRG}} \lambda_B^{b_i} \lambda_Q^{q_i} \lambda_S^{s_i} d_i \delta(m - m_i) \delta(v - v_i)
\]

**PDG hadrons**

\[
\rho_Q(m, v; \lambda_B, \lambda_Q, \lambda_S) = C \nu^\gamma (m - B\nu)^\delta \exp \left\{ \frac{4}{3} [\sigma_Q \nu]^{1/4} (m - B\nu)^{3/4} \right\} \theta(\nu - V_0) \theta(m - B\nu)
\]

**Quark-gluon bags** [J. Kapusta, PRC '81; Gorenstein+, ZPC '84]

Non-overlapping particles \( \rightarrow \) **isobaric (pressure) ensemble**

\[
\hat{Z}(T, s, \lambda_B, \lambda_Q, \lambda_S) = \int_0^\infty Z(T, V, \lambda_B, \lambda_Q, \lambda_S) e^{-sV} \, dV = [s - f(T, s, \lambda_B, \lambda_Q, \lambda_S)]^{-1}
\]

\[
f(T, s, \lambda_B, \lambda_Q, \lambda_S) = \int d\nu \int dm \rho(m, \nu; \lambda_B, \lambda_Q, \lambda_S) e^{-\nu s} \phi(T, m)
\]

The system pressure is \( p = Ts^* \) with \( s^* \) being the **rightmost** singularity of \( \hat{Z} \)
Mechanism for transition to QGP

The isobaric partition function, \( \hat{Z}(T, s, \lambda) = [s - f(T, s, \lambda)]^{-1} \), has

- pole singularity \( s_H = f(T, s_H, \lambda) \) "hadronic" phase
- singularity \( s_B \) in the function \( f(T, s, \lambda) \) due to the exponential spec

\[
\rho_B = T s_B = \frac{\sigma_Q}{3} T^4 - B
\]

MIT bag model EoS for QGP [Chodos+, PRD '74; Baacke, APPB '77]

1st order PT

"collision" of singularities

\( s_H(T_c) = s_B(T_c) \)

2nd order PT

crossover \( s_H(T) > s_B(T) \) at all \( T \)
Crossover transition

Type of transition is determined by exponents $\gamma$ and $\delta$ of bag spectrum

Crossover seen in lattice, realized in model for $\gamma + \delta \geq -3$ and $\delta \geq -7/4$

[ Begun, Gorenstein, W. Greiner, JPG '09 ]

Transcendental equation for pressure:

$$p(T, \lambda_B, \lambda_Q, \lambda_S) = T \sum_{i \in \text{HRG}} d_i \phi(T, m) \lambda_B^b_i \lambda_Q^{q_i} \lambda_S^{s_i} \exp \left(-\frac{m_i p}{4B T} \right)$$

$$+ \frac{C}{\pi} T^{5+4\delta} \left[ \sigma_Q \right]^{\delta+1/2} \left[ B + \sigma_Q T^4 \right]^{3/2} \left( \frac{T}{p - p_B} \right)^{\gamma + \delta + 3} \Gamma \left[ \gamma + \delta + 3, \left( \frac{p - p_B}{T} \right)V_0 \right]$$

Solved numerically

Calculation setup:

$$\gamma = 0, \quad -3 \leq \delta \leq -\frac{1}{2}, \quad B^{1/4} = 250 \text{ MeV}, \quad C = 0.03 \text{ GeV}^{-\delta+2}, \quad V_0 = 4 \text{ fm}^3$$

$$T_H = \left( \frac{3B}{\sigma_Q} \right)^{1/4} \approx 165 \text{ MeV}$$
Thermodynamic functions

Pressure $p/T^4$

- Crossover transition towards bag model EoS
- Dependence on $\delta$ is mild
- Approach to the Stefan-Boltzmann limit is too fast
- Peak in energy density, not seen on the lattice

Energy density $\varepsilon/T^4$

Lattice data from 1309.5258 (Wuppertal-Budapest)
Nature of the transition

Filling fraction \( = \frac{\langle V_{\text{had}} \rangle}{V} \)

- Bags occupy almost whole space at large temperatures
- Strongest changes take place in the vicinity of \( T_H \)
- At \( \delta < -7/4 \) and \( T \to \infty \) whole space — large bags with QGP
Conserved charges susceptibilities

Available from lattice QCD, not considered in this type of model before

\[ \chi_{lmn}^{BSQ} = \frac{\partial^{l+m+n} p / T^4}{\partial (\mu_B / T)^l \partial (\mu_S / T)^m \partial (\mu_Q / T)^n} \]

Qualitatively compatible with lattice QCD
Bag model with massive quarks

Main source of quantitative disagreement comes from inaccuracy of the standard MIT bag model with massless quarks for describing QGP

Quasiparticle models suggest sizable thermal masses of quarks and gluons in high-temperature QGP [Peshier et al., PLB ’94; PRC ’00; PRC ’02]

Heavy-bag model: bag model EoS with non-interacting massive quarks and gluons and the bag constant [Ivanov et al., PRC 72, 025804 (2005)]

Massive quarks and gluons instead of massless ones:

\[
\sigma_Q(T, \lambda_B, \lambda_Q, \lambda_s) = \frac{8}{\pi^2 T^4} \int_0^\infty dk \frac{k^4}{\sqrt{k^2 + m_g^2}} \left[ \exp \left( \frac{\sqrt{k^2 + m_g^2}}{T} \right) - 1 \right]^{-1} \\
+ \sum_{f=u,d,s} \frac{3}{\pi^2 T^4} \int_0^\infty dk \frac{k^4}{\sqrt{k^2 + m_f^2}} \left[ \lambda_f^{-1} \exp \left( \frac{\sqrt{k^2 + m_f^2}}{T} \right) + 1 \right]^{-1} \\
+ \sum_{f=u,d,s} \frac{3}{\pi^2 T^4} \int_0^\infty dk \frac{k^4}{\sqrt{k^2 + m_f^2}} \left[ \lambda_f \exp \left( \frac{\sqrt{k^2 + m_f^2}}{T} \right) + 1 \right]^{-1}
\]
Bag model with massive quarks

Introduction of constituent masses leads to much better description of QGP

Parameters for the crossover model:

\[ m_u = m_d = 300 \text{ MeV}, \quad m_s = 350 \text{ MeV}, \quad m_g = 800 \text{ MeV}, \quad B^{1/4} = 200 \text{ MeV} \]

\[ \gamma = 0, \quad \delta = -2, \quad C = 0.03, \quad V_0 = 8 \text{ fm}^3 \]

\[ T_H \approx 167 \text{ MeV} \]
Hagedorn model: Thermodynamic functions

Pressure $p/T^4$

Energy density $\varepsilon/T^4$

- Semi-quantitative description of lattice data
- Peak in energy density gone!
Hagedorn model: Thermodynamic functions

Trace anomaly \((\varepsilon - 3p)/T^4\)

Speed of sound \(c_s^2 = dp/d\varepsilon\)
Hagedorn model: Susceptibilities

\[
\chi_{lmn}^{BSQ} = \frac{\partial^{l+m+n} \rho / T^4}{\partial (\mu_B / T)^l \partial (\mu_S / T)^m \partial (\mu_Q / T)^n}
\]

Lattice data from 1112.4416 (Wuppertal-Budapest), 1203.0784 (HotQCD)
Hagedorn model: Baryon-strangeness ratio

\[ C_{BS} = -\frac{3\chi_{11}^{BS}}{\chi_2^S} \]

Useful diagnostic of QCD matter

[V. Koch, Majumder, Randrup, PRL 95, 182301 (2005)]

Well consistent with lattice QCD
Higher-order susceptibilities are particularly sensitive probes of the parton-hadron transition and possible remnants of criticality at $\mu_B = 0$.

### Net Baryon Susceptibility

- **$\chi_4^B / \chi_2^B$**
  - **Lattice Data**: 1305.6297 & 1805.04445 (Wuppertal-Budapest), 1708.04897 (HotQCD)
  - **Model Parameters**:
    - Massive quarks
    - $B^{1/4} = 200$ MeV
    - $\gamma = 0$, $\delta = -2$
    - $V_0 = 8$ fm$^3$

### Net Strangeness Susceptibility

- **$\chi_4^S / \chi_2^S$**
  - **Model**
    - $\varepsilon_0 = 4B$
    - $\varepsilon_0 = 16B$

- **Observations**:
  - Drop of $\chi_4^B / \chi_2^B$ caused by repulsive interactions which ensure crossover transition to QGP
  - Peak in $\chi_4^S / \chi_2^S$ is an interplay of the presence of multi-strange hyperons and repulsive interactions