Physics 208

Lecture 1 (Jan 198)

1.1 Reminders about Newtonian Mechanics and Vectors

In the previous lecture, Physics 208, the basic laws of Newtonian mechanics were established. They cannot be derived from any “basic principles” but only by inferences from experience and finaly, systematized laws which have to be verified by experiment.

The basic concepts introduced without further explanation are space and time. The most fundamental entity is a point in space, which is a body so small as to be described by a point in space.

It is further assumed that the motion of point-like objects is described by Euclidean geometry. It happens that motion in space can be described by a vector. A vector in space can be described by an arbitrary fixed point in space which is its origin, and a direction (arrows) which points to the end of the vector. The motion of the particle is described according to it by a vector, e.g., for a particle in a plane or in space.

A vector can be decomposed into components with respect to this coordinate system (or basis vectors):

\[
\vec{r} = x \hat{i}_x + y \hat{i}_y \quad \text{(plane, 2D)}
\]

\[
\vec{r} = x \hat{i}_x + y \hat{i}_y + z \hat{i}_z \quad \text{(space, 3D)}
\]
Vectors can be added as described in the following way: Suppose one has a point particle located at \( P_1 \) and is then shifted to another position \( P_2 \). For simplicity's sake we work in a plane. Then we have:

\[
\vec{r}_2 = \vec{r}_1 + \vec{d}
\]

Note that \( d_y < 0 \) because the arrow is pointing downwards, in opposition to \( \vec{e}_y \).

Thus we obviously have:

\[
\vec{r}_2 = \vec{r}_1 + \vec{d}
\]

Now we have

\[
\begin{align*}
\vec{r}_1 &= (x_1, y_1) \\
\vec{d} &= (d_x, d_y)
\end{align*}
\]

Then the formula above reads

\[
\vec{r}_2 = (x_2, y_2) = \vec{r}_1 + \vec{d} = (x_1 + d_x, y_1 + d_y)
\]

Sometimes one works the vectors in column form by first writing the components, e.g.,

\[
\vec{r}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \text{ and }
\vec{d} = \begin{pmatrix} d_x \\ d_y \end{pmatrix}
\]

Thus the formula above reads

\[
\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + d_x \\ y_1 + d_y \end{pmatrix}
\]

Voters can also be multiplied by real numbers. If the

Voters are positive, the resulting vector points in the same

Voters, but the length is multiplied by the same.
The motion of a point is described by the vector \( \mathbf{r} \) as a function of time:

\[
\mathbf{r} = \mathbf{r}(t) = x(t) \mathbf{i}_x + y(t) \mathbf{i}_y + z(t) \mathbf{i}_z
\]

and one can use algebra and analysis (calculus) to describe its motion in this path.

The velocity is given by the time derivative:

\[
\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \mathbf{r}'(t) = \dot{x}(t) \mathbf{i}_x + \dot{y}(t) \mathbf{i}_y + \dot{z}(t) \mathbf{i}_z
\]
and the acceleration by
\[ \ddot{a}(t) = \frac{d\ddot{V}(t)}{dt} = \frac{d^2 \ddot{r}(t)}{dt^2} = \dddot{r}(t) = \dddot{x}(t) \]

Then Newton's laws read (in an inertial frame of reference):

1. A particle moves with constant velocity if no force is changing it.

2. The force is proportional to the acceleration of the point particle. The proportionality constant is called mass of the particle:
\[ \mathbf{F} = m \ddot{\mathbf{r}} \]

Force and mass are not further defined by simple concepts but on basic motions of mechanics! An important example of a force is gravity. Two point masses are 1 unit apart. According to Newton's law of gravitation:
\[ \left| \mathbf{F}_{12} \right| = G \frac{m_1 m_2}{x^2} \quad \text{with} \quad x = 1 \text{ unit} \]

or some
\[ \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}_1 - \mathbf{r}_2 \]
\[ \left| \mathbf{F}_{12} \right| = G \frac{m_1 m_2}{\left| \mathbf{r}_1 - \mathbf{r}_2 \right|^2} \]

The force is a vector and directed along \( \mathbf{r}_1 \), i.e.
\[ \mathbf{F}_{12} = \mathbf{F} \frac{x^2}{x} = \mathbf{F} \frac{x}{x^2} \]
\[ = \mathbf{F} \frac{m_1 m_2}{\left| \mathbf{r}_1 - \mathbf{r}_2 \right|^2} \frac{\mathbf{r}_1 - \mathbf{r}_2}{\left| \mathbf{r}_1 - \mathbf{r}_2 \right|} \]
In the same way one obtains the force exerted by the point particle 2 on point particle 1:

\[ F_{21} = \frac{G m_1 m_2}{r_{21}^2} \left( \frac{\hat{r}_{21}}{r_{21}} - \hat{r}_{21} \right) = -\hat{r}_{12} \]

That's not only true for the gravitational force, but a general law, known as Newton's 3rd law.

(3) If particle 1 exerts a force \( \vec{F}_{12} \) on particle 2, then particle 2 exerts a force \( \vec{F}_{21} \) which is equal in magnitude but directed in the opposite way:

\[ \vec{F}_{21} = -\hat{r}_{12} \]
1.2 Electric charge and Coulomb's Law

Ancient Greeks, about 2500 years ago discovered that rubbing an object
would cause certain other materials. Some amber is called
"elektron." Such phenomena are called "electric.
To describe such electric phenomena one introduces another basic
quantity (which cannot be explained by simple means), today
called "electric charge."

In the last 18th century Coulomb and Priestly found the
law of force for electrically charged point particles to be
similar to Newton's law of gravitation

\[ F_E = \frac{1}{4 \pi \varepsilon_0} \frac{|q_1 q_2|}{x^2} \]

where \( q_1 \) and \( q_2 \) are the charges of the point particles, and \( x \)
the distance between them. \( \varepsilon_0 \) is a constant which determines
the unit of charge.

Thus, many systems of units in use. In this lecture we
shall use the "standard" SI (System International de
Unites) system. The SI system and its notation will be
used throughout this course.

The basic unit for charge in this system is called the coulomb.

In the SI system the constant \( \frac{1}{4 \pi \varepsilon_0} \) is taken as \( 1 \), where \( \varepsilon_0 \)
is called "permittivity of free space." Then Coulomb's law reads

\[ F_E = \frac{|q_1 q_2|}{4 \pi \varepsilon_0 x^2} \]
The value is
\[ x = \frac{1}{4536} \approx 9.10^3 \frac{Nm}{m^2} \]

Recall that \( N \) is the mass unit of force
\[ 1N = 1 \frac{kg \cdot m}{s^2} \]
(known as newton)

\( m \) is the unit of length (meter) and \( g \) the unit of mass
and \( \frac{m}{s^2} \) the unit of time (second).

Charges can appear with both positive and negative
and it is found that charges of

like signs repel

unlike signs attract

Each other and the direction of the force is along the line
from one charge to the other, in particular analogy to Newton's law
for the gravitational force.

Thus we know
\[ \frac{q_1 q_2}{4 \pi \varepsilon_0} \]
\[ \frac{-x}{x^2} \]
with
\[ x = \frac{x}{x} \]

The net electric field in the region with charges \( q_1 \) and \( q_2 \) might have. Indeed, if \( q_1 \) and
\( q_2 \) have like signs, \( q_1, q_2 > 0 \), and the force is repulsive while
\( q_1 \) have like signs, \( q_1, q_2 > 0 \), and the force is repulsive while
of the same magnitude, the force is attraction.
As suggested in the book, it can be advantageous to define the symbol \( q \) as position and with \(-q\) for negative charge. Now we can calculate the electric forces of point particles and use them in force diagrams as needed.

**Example**

\[ F_E = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{(2L \sin \theta)^2} \]

For the calculation of the left charge:

\[ \sum F = m a = 0 \]

Now \( \sum F = 0 \):

\[ -F_E + T \sin \theta = 0 \Rightarrow F_E = T \sin \theta \]

**x-direction:** \[ -F_E + T \cos \theta = 0 \Rightarrow T = \frac{mg}{\cos \theta} \]

**y-direction:** \[ T \sin \theta - Mg = 0 \Rightarrow T = \frac{mg}{\sin \theta} \]

Thus we find:

\[ F_E = \frac{mg}{\cos \theta} \sin \theta = mg \tan \theta \]

\[ = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{(2L \sin \theta)^2} \]
Suppose we know \( q \) and measure the angle \( \theta \), we can determine 
\[
    \sin \theta = \frac{q}{\sqrt{q^2 - \left(2L \sin \theta \right)^2}}
\]

To get an idea about orders of magnitude, suppose \( q = 1 \), \( L = 1 \) m, and \( \theta = 30^\circ \). Then 
\[
    \sin \theta = \frac{1}{\sqrt{1 - \left(2 \times \frac{1}{2} \right)^2 \times \frac{1}{2}}} 
    \approx 0.577 
    \quad N \approx 15.6 \times 10^3 \text{ N} 
    \approx 3.5 \times 10^5 \text{ lbs}
\]
(\( \approx 3.5 \times 10^5 \) lbs)

That shows that 1C is a rather large charge.

Nowadays we know that by warming e.g., water with e.g., heat, we can produce some small charges of a few or a few hundred millivolts, each of order basic charge of
\[
    q_{\text{electron}} = -e = -1.6 \times 10^{-19} \text{ C}
\]
The mass of an electron is
\[
    m_{\text{electron}} = 9.1 \times 10^{-31} \text{ kg}
\]

Example

Compare electrical force between 2 electrons with an 10^{-3} m apart.

The gravitational force between the
\[
    F_g = \frac{G m_1^2}{x^2} \approx 2.3 \times 10^{-12} \text{ N}
\]

The Coulomb force is
\[
    F_e = k \frac{q_1 q_2}{x^2} \approx 5.4 \times 10^{-57} \text{ N}
\]

\[
    \frac{F_e}{F_g} = \frac{9 \times 10^{-9}}{4.3 \times 10^{-6}} \approx 4.3 \times 10^2
\]

In this sense, the electric force is much larger than the gravitational force.
If we have only point particles, the Coulomb forces simply add (as vectors!). In such a case we need to use spherical coordinates, especially if the distance between the charges is not large. It is important to notice that this class of problems need to be solved individually for each object, where the charge distribution can be changed with time for charged objects, and by the presence of other charges.

**Example**

**Total force on the 3rd charge:**

\[ \vec{F}_{33} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_3}{r_{13}^3} \hat{r}_{13} \]

\[ \vec{F}_{23} = \frac{1}{4\pi\varepsilon_0} \frac{q_2 q_3}{r_{23}^3} \hat{r}_{23} \]

Here all the \( q \)'s are taken positive. The 2nd charge is thus away from \(-q_2\).

Decomposition into components: For all charges lie in the xy-plane, we can work with plane vectors:

\[ \vec{r}_{13} = \omega_3 \sin \theta \hat{i}_x + \cos \theta \hat{i}_y \]

\[ \vec{r}_{23} = \omega_4 \sin \theta \hat{i}_x - \cos \theta \hat{i}_y \]

\[ \vec{F}_{\text{total}} = \frac{1}{4\pi\varepsilon_0} \left[ \left( \frac{q_1 q_3}{r_{13}^2} \cos \theta + \frac{q_2 q_3}{r_{23}^2} \cos \phi \right) \hat{i}_x + \right. \]

\[ + \left( \frac{q_1 q_3}{r_{13}^2} \sin \theta - \frac{q_2 q_3}{r_{23}^2} \sin \phi \right) \hat{i}_y \]
Lecture 3-4 (Jan 24 - 26)

2.1 The Electric Field

Suppose we have any number of charges $q_1, \ldots, q_m$ placed arbitrarily at spatial points $\mathbf{p}_1, \ldots, \mathbf{p}_m$.

Now we ask what force is exerted on a particle $q_0$ if placed at $\mathbf{p}_0$. From Coulomb's law, we know that if $q_0$ is so small that the charges $q_1, \ldots, q_m$ keep at the above five places:

$$\mathbf{E}_{\text{total}}(\mathbf{r}) = \sum_{i=1}^{m} \frac{q_i}{4\pi\varepsilon_0 \left| \mathbf{r} - \mathbf{r}_i \right|^3} \mathbf{r} - \mathbf{r}_i$$

These forces are:

$$\frac{q_0 q_1}{4\pi\varepsilon_0 \left( r_0^2 - r_1^2 \right)^{3/2}}$$

$$\frac{q_0 q_2}{4\pi\varepsilon_0 \left( r_0^2 - r_2^2 \right)^{3/2}}$$

$$\vdots$$

$$\frac{q_0 q_m}{4\pi\varepsilon_0 \left( r_0^2 - r_m^2 \right)^{3/2}}$$

$$\mathbf{E}_{\text{total}}(\mathbf{r}) = \sum_{i=1}^{m} \frac{q_i}{4\pi\varepsilon_0 \left| \mathbf{r} - \mathbf{r}_i \right|^3} \mathbf{r} - \mathbf{r}_i$$
Thus we can define, by measuring the force on the charge $q_0$ in the field of $q_1\rightarrow q_0$ a quantity which depends only on the charges $q_1, \ldots, q_n$ which we call the electric field

$$E^0(r_0) = \sum_{n=1}^{N} \frac{q_n}{4\pi\epsilon_0 \left| r_0 - r_n \right|^2}$$

The important concept behind this definition is that we can in effect sum the actions of all charges $q_n$ located in $\mathbb{R}^3$ not only as a force acting at a distant point as the sum of a physical object spread out in all space, namely the electric field, $E$, which causes the force of all the other charges, $q_n$, on another charge, $q_0$, which is small enough not to disturb the other charges, $q_n$,

$$F = q_0 \cdot E(r_0)$$

if it is located at $r_0$. In the following we call such a charge a test charge (for convenience). This important concept is due to Michael Faraday, who investigated experimentally a large number of electromagneto optical phenomena.

To calculate an electric field produced by point sources, first calculate the Coulomb force on a test charge, $q_0$, located at $r_0$ and finally divide by $q_0$ (see the equation above).
Example 1

Take one point charge \( q \). Then we can put the origin of the coordinate system at this charge. The electric field on the test charge \( q_0 \) located at \( r \) is the

\[
E(r) = \frac{kq_0}{r^2} \quad \text{or} \quad E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}
\]

Example 2

\[ \text{NOTE: y axis points down and clockwise} \]

What is the electric field along the \( x \) axis?  

Use our work with components! (Note also the method in the last book assigning angles and trig functions.)

\[
E(x, 0) = \frac{1}{4\pi\varepsilon_0} \left[ \frac{a}{(x^2 + a^2)^{3/2}} \left( \frac{x^3}{x^2 + a^2} \right) + \frac{-q}{(x^2 + a^2)^{3/2}} \left( \frac{x^2}{x^2 + a^2} \right) \right]
\]
2.2 Motion of a test particle in an electric field

If the charge \( q \) is sufficiently small compared to all charges involved in creating the electric field (such that we can consider the locations of these charges as unaffected by the test charge), we can calculate its motion by using Newton’s law:

\[
ma = F = qE(x) \]

If we know the initial conditions:

\[
\mathbf{x}(0) = \mathbf{x}_0, \quad \dot{\mathbf{x}}(0) = \mathbf{v}_0
\]

by solving the set of differential equations:

\[
m\ddot{\mathbf{x}} = q\mathbf{E}(\mathbf{x})
\]

for the components of \( \mathbf{x} \).

Example 1: Motion of a charge in constant electric field

Let \( \mathbf{E} = \text{const} \). Then we may choose the coordinate system such that \( \mathbf{E} \) points in positive \( x \)-direction.

\[
0 \rightarrow \mathbf{r} = q \mathbf{E} = q \mathbf{E} \mathbf{e_x}
\]
Then
\[ m \ddot{x} = qE \]
\[ m \ddot{y} = 0 \]
\[ m \ddot{z} = 0 \]

The particle initially may be at rest in the coordinate origin:
\[ x(0) = y(0) = z(0) = 0 \]
\[ v_x(0) = v_y(0) = v_z(0) = 0 \]

Since \( E = \text{const} \) it is easy to solve the equation for \( x \):
\[ x = \frac{q}{m} E \Rightarrow \dot{x} = \frac{q}{m} E t + x_0 = \frac{q}{m} E t \]

Integrating once more:
\[ x = \int_0^t \dot{x}(t') dt' = \frac{q}{m} E \int_0^t t' dt' = \frac{q}{m} E \frac{t^2}{2} \]

\[ x(t) = \frac{qE}{2m} \frac{t^2}{2} \]

The other 2 components are trivially found:
\[ y(t) = z(t) = 0 \]
3. The electric potential (will of fun (29))

3.1 Review on conservative forces

In mechanics, we have learnt about the concept of energy. Here, when 0 is considered, we start with Newton's 2nd law:

\[ M \frac{d^2 \theta}{dt^2} = F \]

Now suppose we have solved the equations of motion, i.e., we know the trajectory of the particle \( C: x = \vec{x} \) (16)

Then we multiply the Eqn with \( \frac{d}{dt} \)

\[ M \frac{d^2 \vec{x}}{dt^2} = \vec{F} \]

where we used the dot product. In cartesian coordinates, it's defined by

\[ \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \]

which is a scalar (a scalar).

Now it is easy to see that we can work

\[ M \frac{d^2 \vec{x}}{dt^2} = \frac{1}{2} \frac{d}{dt} \left( \frac{\vec{v}^2}{2} \right) \]

To prove this, first use the product rule for differentiation.

Thus we can write

\[ \frac{d}{dt} \left[ \frac{1}{2} \vec{v}^2 \right] = \vec{v} \cdot \vec{F} \]

Now we can integrate this equation from \( t_1 \) to \( t_2 \).
\[ \int_{t_1}^{t_2} \left( \frac{m}{2} \dot{r}^2 \right) dt = \frac{m}{2} \left[ \dot{r}^2(t_2) - \dot{r}^2(t_1) \right] \]

The RHS is the work done on the particle along its trajectory:
\[ W_c = \int_{t_1}^{t_2} \dot{r} \cdot \vec{F} \, dt \]

The change in kinetic energy is equal to the work done on the particle moving along its trajectory.

**Conservative Forces**

It helps simplify the notation if \( \vec{F} \) depends only on the position of the particle, not its velocity (i.e., magnetic force) and if the work done on the particle is independent on the curve connecting \( \vec{r}_1 \) and \( \vec{r}_2 \):

\[ W_c = W(\vec{r}_1, \vec{r}_2) = \int_{t_1}^{t_2} dt \frac{d\vec{r}^2}{dt} \vec{F} \cdot d\vec{r} \]

\[ = \int_{C} d\vec{r} \cdot \vec{F}(\vec{r}) \]

\[ = \int_{C_1} d\vec{r} \cdot \vec{F}(\vec{r}) \]
We can rephrase this also in the way that
\[ W_c = 0 \] for all closed curves \( C \).

The point to
\[ \int f(x) \text{ as a constant over the region we can reconstruct the function} \]
\[ \text{the work as a function of its end point} \]
\[ W(x) = \int_{C_0} \nabla \cdot \vec{F} \cdot dr \]

When \( C \) is an arbitrary arc connecting some arbitrary fixed point \( C_0 \) with the end point \( x \).

Now we can ask about the derivative of \( W \):

\[ \frac{dW}{dx} = \lim_{\Delta x \to 0} \frac{W(x + \Delta x, \vec{x}) - W(x, \vec{x})}{\Delta x} \]

\[ = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[ \int_{\Delta C} \nabla \cdot \vec{F} \cdot d\vec{r} - \int_{C_0} \nabla \cdot \vec{F} \cdot d\vec{r} \right] \]

\[ = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{C_0+C'} \nabla \cdot \vec{F} \cdot d\vec{r} \]

Now we parametrize \( C' \) simply as \( x \):

\[ C': \vec{r} = x \vec{x} \]
Thus
\[
\frac{\Delta W}{\Delta X} = \lim_{\Delta X \to 0} \frac{1}{\Delta X} \int_0^{\Delta X} dx \ i_x \ F_x (\vec{r} + x \ i_x)
\]
\[= \lim_{\Delta X \to 0} \frac{1}{\Delta X} \int_0^{\Delta X} dx \ F_x (\vec{r} + x \ i_x) \ dx
\]
If now \(\Delta X\) is very small, we can work
\[F_x (\vec{r} + x \ i_x) \approx F_x (\vec{r})\]
and then
\[
\frac{\Delta W}{\Delta X} = \lim_{\Delta X \to 0} \frac{1}{\Delta X} \int_0^{\Delta X} dx \ F_x (\vec{r}) = \lim_{\Delta X \to 0} \frac{F_x (\vec{r})}{\Delta X} \left[ x \right]_{x=0}^{x=\Delta X}
\]
\[= F_x (\vec{r}) \lim_{\Delta X \to 0} \frac{\Delta X}{\Delta X} = F_x (\vec{r})
\]
In the same way we find
\[F_y (\vec{r}) = \frac{\Delta W (\vec{r})}{\Delta y}
\]
\[F_z (\vec{r}) = \frac{\Delta W (\vec{r})}{\Delta z}
\]
Now we come back to our energy formula
\[E_{\text{kin}2} - E_{\text{kin}1} = \int d\vec{r}^2 \ F (\vec{r}) = U(\vec{r}_1) - U(\vec{r}_2)
\]
Indeed:
\[U(\vec{r}_1) = \int d\vec{r}^2 \ F (\vec{r})\]
\[U(\vec{r}_2) = \int d\vec{r}^2 \ F (\vec{r})\]
So we have if we start from the origin of \( C_1 \) to \( -C_1 \):

\[
W(\vec{r}_2) - W(\vec{r}_1) = \int_{C_2-C_1} d\vec{r} \cdot \vec{F}(\vec{r})
\]

On the other hand, since \( \vec{F} \) is conservative, the work done on the particle along its trajectory:

\[
W(\vec{r}_2) - W(\vec{r}_1) = \int_{C} d\vec{r} \cdot \vec{F}(\vec{r}) = W_C
\]

Thus we have

\[
E_{\text{kin}}_{12} - E_{\text{kin}}_{11} = W(\vec{r}_2) - W(\vec{r}_1)
\]

or

\[
E_{\text{kin}}_{12} - W(\vec{r}_2) = E_{\text{kin}}_{11} - W(\vec{r}_1)
\]

This holds for all \( \vec{r}_2 \) if we keep the initial point.

Thus holds for all \( \vec{r}_2 \) if we keep the initial point.

\[
E_{\text{pot}}_{12} = -W(\vec{r}_2) = W(\vec{r}_1)
\]

We have the law of conservation of energy:

\[
E_{\text{kin}} + E_{\text{pot}} = \text{const.} \quad \text{(Conservation law: The total energy stays constant.)}
\]
\[ m \frac{\partial^2 \mathbf{r}}{\partial t^2} + U(r) = \text{const.} \]

We can easily prove this by using the kinetic energy:

\[ \frac{d}{dt} \left[ \frac{m}{2} \frac{\partial^2 \mathbf{r}}{\partial t^2} \right] = \frac{m}{2} \frac{\partial}{\partial t} \frac{\partial^2 \mathbf{r}}{\partial t^2} = \frac{m}{2} \mathbf{a} \cdot \mathbf{a} = m \mathbf{a}^2 \]

\[ \frac{d}{dr} \left[ U(r) \right] = \frac{\partial U}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial U}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial U}{\partial z} \frac{\partial z}{\partial r} \]

But now we have seen that

\[ \mathbf{F} = \frac{\partial U}{\partial \mathbf{r}} = -\frac{\partial U}{\partial \mathbf{r}} \Delta \mathbf{r} \]

So we have

\[ \frac{d}{dr} U(r) = - \left( \mathbf{F} \cdot \mathbf{r} + \mathbf{F}_y \frac{\partial y}{\partial r} + \mathbf{F}_z \frac{\partial z}{\partial r} \right) \]

\[ = - \mathbf{F} \cdot \mathbf{r} \quad (= - P) \]

\[ \Rightarrow \quad \frac{d}{dt} \left[ \frac{m}{2} \mathbf{r}^2 + U(r) \right] = m \mathbf{a} \cdot \mathbf{a} - \mathbf{F} \cdot \mathbf{r} \]

\[ = \mathbf{r} \cdot \frac{\partial^2 \mathbf{r}}{\partial t^2} = 0 \]

because of Newton's 2nd law.

On the other hand, if this looks a bit much we'll

\[ \mathbf{F}(r) = \mathbf{F}_r \left( - \frac{\partial U}{\partial \mathbf{r}} \right) \]
\[ \int dr^2 \cdot \vec{F}(\vec{r}) = \int_{t_1}^{t_2} dt \frac{d^2 \vec{r}}{dt^2} \cdot \vec{F}(\vec{r}(t)) \]

\[ = -\int_{t_1}^{t_2} dt \left[ \frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt} + \frac{d\vec{y}}{dt} \cdot \frac{d\vec{y}}{dt} + \frac{d\vec{z}}{dt} \cdot \frac{d\vec{z}}{dt} \right] \]

\[ = -\int_{t_1}^{t_2} dt \frac{d}{dt} U[\vec{r}(t)] \]

\[ = -U[\vec{r}(t)] \bigg|_{t_1}^{t_2} = -\left[ U(\vec{r}_2) - U(\vec{r}_1) \right] \]

Independent of the curve \( C \) \( \Rightarrow \) \( \vec{F} \) is conservative

**Examples**

1) Constant force e.g. gravity near the Earth
   \[ \vec{F} = -mg \hat{y} \]

**Fund potential:** If \( \vec{F} \) exists we can choose any curve \( C \) resulting in the same work
   \[ \vec{F} \text{ over any loop } C = C_1 + C_2 \] (see below)
C1: \( \hat{z} = x \) with \( x \in (0,1) \)  

C2: \( \hat{z} = y + \hat{z} \)

\( \hat{z} \in (0,1) \)

\[ U(\hat{z}) = - \int_{x \in 0} dV \hat{z} \hat{z} + y \int_{x \in 0} dV \hat{z} \hat{z} \]

= \int_{x \in 0} dV \hat{z} \hat{z} (-mg \hat{z}) - \int_{x \in 0} dV \hat{z} \hat{z} (-mg \hat{z})

= mg \hat{z}

\text{Energie: } E = \frac{m}{2} \hat{\dot{z}}^2 + mg \hat{z}.

\text{Check: } F_x = - \frac{\partial U}{\partial x} = 0 \checkmark

F_y = - \frac{\partial U}{\partial y} = -mg \checkmark

\( P \) is really conserved

\( \Rightarrow E = \text{const. of motion} \)
Examples for potentials of forces (Jan 30)

(a) Constant force

\[ \vec{F} = F \vec{m} \text{ with } m, F = \text{const}. \]

How to find the potential \( \Phi \) of a force. Two methods:

(1) Take an arbitrary path from a fixed point to \( \vec{r} \):

\[ \vec{r} = \vec{r}_{\text{fix}} \]

\[ \hat{r} = \frac{\vec{r}}{|\vec{r}|} \text{ where } |\vec{r}| = 1 \]

Take an arbitrary path from the origin to \( \vec{r} \) and compute
\[ \vec{F} = -\vec{g} \]

\[ \vec{F} = -\left( \frac{\partial U}{\partial x} \hat{x} + \frac{\partial U}{\partial y} \hat{y} + \frac{\partial U}{\partial z} \hat{z} \right) \]

The last idea is to assume a straight line:

\( \vec{C} : \Phi (x) = 2x, x \in [0, 1] \)

\[ d\Phi = \frac{d\vec{r} \cdot \vec{F}}{d\vec{r}} = \vec{F} \cdot d\vec{r} \]

\[ U = -\Phi (x) = -2x \int_0^x d\vec{r} \]

\[ \text{Check:} \]

\[ -\frac{\partial U}{\partial x} = F_x \]

(2)
(P) solve partial DES.

\[ F = -\frac{\partial u}{\partial x} \]

Since \( F \) is constant, that's easy.

\[ \frac{\partial u}{\partial x} = F \Rightarrow u = -Fx + u_1(x,t) \]

\[ \frac{\partial u_1}{\partial y} = -F \Rightarrow u_1 = -Fy + u_2(x,t) \]

\[ \frac{\partial u_2}{\partial t} = -F \Rightarrow u_2 = -Ft + \text{const.} \]

\[ u(\vec{r}) = -\left( Fx + \ldots \right) = -F \vec{r} + \text{const.} \]

\[ \text{const. is undetermined and physically insignificant.} \]

In method (P) it's hidden in the arbitrariness of the choice of the initial point.

(1) Force of a spring

\[ F = -D \frac{\partial^2 u}{\partial x^2} \]

The 2nd method is easiest. We can assume \( u = u(x) \)

because there's only an \( x \)-directed force.

So we obtain

\[ \frac{\partial u}{\partial x} = -Dx \Rightarrow u(x) = -\frac{D}{2} x^2 + \text{const.} \]
Constant $E^2$ field

That we have already solved just a pop a pop

$F^2 = q E^2 = w_0 f$

$U = -q E \cdot \mathbf{x}$

$\Rightarrow$

was how to solve

(9) Electrostatic system

The statement in the book that the potential cannot be determined in electrostatic coordinates is incorrect!

This is the proof why

\[ F = \frac{1}{4 \pi \varepsilon_0} \frac{q q_0}{r^3} = \nabla \left( \frac{q^2}{r^3} \right) \quad \text{(when } q = \frac{1}{4 \pi \varepsilon_0} q q_0 = \text{const)} \]

with the method (9) it's easy!

\[ \frac{\partial U}{\partial x} = -F_x = -\frac{q}{r^3} \frac{x}{(x^2 + y^2 + z^2)^{3/2}} \]

Thus we calculate by $x$

\[ U = -\nabla \int dx \left( -\frac{x}{(x^2 + y^2 + z^2)^{3/2}} \right) + U_1(x, y, z) \]

Suggestion when:

$\nabla = x^2 \Rightarrow \int dx = \frac{1}{2} \int du$

\[ U = -\frac{1}{2} \int \frac{du}{(u^2 + y^2 + z^2)^{3/2}} + U_1(x, y, z) \]
\[
\begin{align*}
U &= -\frac{A}{2} \int du \left( u + y^2 + z^2 \right)^{-\frac{3}{2}} + u_1(y) \\
&= -\frac{A}{2} \left( -\frac{3}{2} + 1 \right) (x^2 + y^2 + z^2)^{-\frac{3}{2} + 1} \\
&= A \left( x^2 + y^2 + z^2 \right)^{-\frac{3}{2}} = \frac{A}{r} + u_1(y) \\
U &= \frac{1}{\sqrt{4\pi\varepsilon_0}} + u_1(y)
\end{align*}
\]

\[
\frac{\partial U}{\partial y} = -\frac{1}{\sqrt{4\pi\varepsilon_0}} \frac{4\pi\varepsilon_0}{r^2} \frac{d}{dy} \left( y \right) + \frac{\partial u_1(y)}{\partial y}
\]

\[
= -\frac{4\pi\varepsilon_0}{r^3} y + \frac{\partial u_1(y)}{\partial y}
\]

\[
= -F_y + \frac{\partial u_1}{\partial y}
\]

\(\Rightarrow\) I can choose \(\frac{\partial u_1}{\partial y} = 0\) \(\Rightarrow\) \(u_1 = u_2(x)\)

In the same way: \(\frac{du}{dt} = -F_x + \frac{du_2}{dt} \Rightarrow \frac{du_2}{dt} = 0\)

\(\Rightarrow\) \(U = \frac{1}{\sqrt{4\pi\varepsilon_0}} \frac{4\pi\varepsilon_0}{r} \)

\(\Rightarrow\) \(U = \frac{1}{\sqrt{4\pi\varepsilon_0}} \frac{4\pi\varepsilon_0}{r}\)
Use polar coordinates. I cannot choose the origin as starting point for my integral because $P$ is surrounded at $r=0$.

\[ P = \sqrt{(r \cos \theta)^2 + (r \sin \theta)^2} \]

Unit vectors in direction of the coordinate lines are given by:

\[ \hat{r} = \frac{\partial P}{\partial r} \frac{1}{\sqrt{\left(\frac{\partial P}{\partial r}\right)^2}} \]

\[ \hat{\theta} = \frac{\partial P}{\partial \theta} \frac{1}{\sqrt{\left(\frac{\partial P}{\partial r}\right)^2}} \]

\[ \hat{\theta} = \cos \theta \hat{x} + \sin \theta \hat{y} \]

\[ \frac{\partial P}{\partial x} = \sqrt{\cos^2 \theta + \sin^2 \theta} \]

\[ \left| \frac{\partial P}{\partial x} \right| = 1 \]

\[ \hat{v} = \cos \theta \hat{x} + \sin \theta \hat{y} \]

\[ \hat{g} = \frac{\partial^2 P}{\partial \theta^2} \frac{1}{\left| \frac{\partial P}{\partial \theta} \right|} \]
\frac{d^2\varphi}{ds^2} = \sqrt{-\sin\theta \frac{d\varphi}{dx} + \cos\theta \frac{d\varphi}{dy}}
\frac{d^2\varphi}{ds^2} = \sqrt{-\sin\theta \frac{d\varphi}{dx} + \cos\theta \frac{d\varphi}{dy}}
\Rightarrow \frac{d\varphi}{ds} = \sqrt{-\sin\theta \frac{d\varphi}{dx} + \cos\theta \frac{d\varphi}{dy}}
Note that
\frac{d\varphi}{dx} \frac{d\varphi}{dy} = 0 \text{ as it should be since the vector of } \frac{d\varphi}{dx}
\text{ is always \perp to the tangent at the point it touches the circle.}
The law in the new coordinates is
\tilde{F} = \frac{1}{4\pi\varepsilon_0} \frac{\varphi_0}{r^2}
and now the path is easily chosen. As a skier point
\text{ to } r = r_0, \varphi = 0:

The path consists of two parts:
\begin{align*}
C_1: & \tilde{r}_1(u) = (r_0 + \varphi_0 \cos u, \varphi_0 \sin u) \text{ with } u \in [0, \pi] \\
& d\tilde{r}^2 = \frac{d\varphi^2}{ds} du = r_0 \varphi_0^2 \\
& |d\tilde{r}| = r_0 \varphi_0 \sin u \\
C_2: & \tilde{r}_2(s) = \tilde{r}(\pi) \text{ with } s \in [0, \pi] \\
& d\tilde{r}^2 = \frac{d\varphi^2}{ds} ds = \varphi_0 \varphi_1 ds
\end{align*}
\[ U = -\int \frac{d^3 \psi}{(2\pi)^3} \phi \frac{\partial^2}{\partial \vec{r}^2} \psi (\vec{r}) - \int_0^\infty d\xi \xi (\xi) \frac{\xi}{8} \frac{\partial^2}{\partial \vec{r}^2} \psi (\vec{r}) \]

\[ = -\int_0^\infty d\xi \xi \frac{\partial}{\partial \xi} \left[ \phi \frac{\partial}{\partial \xi} \psi (\xi) \right] = 0 \]

\[ = -\int_0^\infty d\xi \frac{A}{8} \left[ \frac{\partial}{\partial \xi} \xi \right] = \frac{A}{8} \left( \frac{1}{\xi} - \frac{1}{\xi_0} \right) \]

This is the same result as with the other method with

This is the same result as with the other method with a constant \(-\frac{A}{8}\) which is irrelevant. Usually, one chooses a constant \(-\frac{A}{8}\) which is irrelevant. Usually, one chooses a constant \(-\frac{A}{8}\) which is irrelevant. Usually, one chooses a constant \(-\frac{A}{8}\) which is irrelevant.

Such a potential is consistent with \(\xi \to \infty\) i.e. one has \(\psi_0 \to \infty\)

\[ U_{\xi_0 \to \infty} = \frac{A}{\sqrt{\xi}} = \frac{1}{4\pi \varepsilon_0} \frac{q^2}{\varepsilon_0} \]
\[ P = \frac{\partial}{\partial x} \left( \begin{array}{c} \partial \\ \partial y \end{array} \right) \]

Note that we must be careful with our choice of coordinates to calculate forces from the potential in general coordinates. Consider an infinitesimal displacement $\delta x$ and $\delta y$.

Using Cartesian coordinates, we get the force by the partial derivative:

\[ F = -\nabla U = -\frac{\partial U}{\partial x} \hat{x} - \frac{\partial U}{\partial y} \hat{y} \]

In Cartesian coordinates, we get the force by the partial derivative:

\[ F = -\nabla U = -\frac{\partial U}{\partial x} \hat{x} - \frac{\partial U}{\partial y} \hat{y} \]

We can write this in a coordinate-independent form:

\[ \nabla U = -\frac{\partial}{\partial x} \hat{x} - \frac{\partial}{\partial y} \hat{y} \]

Since $x^2 + y^2 = \rho$, the components of $\nabla U$ in polar coordinates are:

\[ \frac{\partial U}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial U}{\partial \theta} \hat{\theta} \]

Now suppose we have a function $U$ in terms of polar coordinates.

We have the following equations:

\[ dU = \frac{\partial U}{\partial \rho} d\rho + \frac{1}{\rho} \frac{\partial U}{\partial \theta} d\theta \]

Now we define:

\[ g_r = \left| \frac{\partial \rho}{\partial x} \right| \quad \text{and} \quad g_\theta = \left| \frac{\partial \rho}{\partial \theta} \right| \]

So we have by definition:

\[ \frac{\partial^2}{\partial x^2} = g_r \frac{\partial^2}{\partial \rho^2} + g_\theta \frac{\partial^2}{\partial \theta^2} \]
In the lectures, we have seen that

\[ g_r = 1 \text{ and } g_\theta = r \]

Thus, we have

\[ \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} = -F_r \frac{\partial r}{\partial \theta} \left( g_r \frac{\partial v}{\partial r} + g_\theta \frac{\partial v}{\partial \theta} \right) \]

Now, sum

\[ \frac{\partial u}{\partial r} + \frac{\partial v}{\partial \theta} = 1 \text{ and } \frac{\partial r}{\partial \theta} g_\theta = 0 \]

We have

\[ \frac{\partial u}{\partial r} = F_r \frac{\partial v}{\partial r} + F_\theta \frac{\partial v}{\partial \theta} \]

\[ \Rightarrow \frac{\partial u}{\partial r} = F_r \text{ and } \frac{\partial v}{\partial \theta} = F_\theta \]

So we have

\[ \frac{\partial u}{\partial r} + \frac{\partial v}{\partial \theta} = -F_r \frac{\partial v}{\partial r} - r F_\theta d \theta \]

Now, we can set \( d \theta = 0 \) and \( rv \to 0 \)

\[ \Rightarrow \frac{\partial u}{\partial r} = -F_r \frac{\partial v}{\partial r} \]

\[ \Rightarrow \frac{\partial v}{\partial \theta} = F_\theta \]

or we can set \( dv = 0 \) and \( d \theta \to 0 \)

\[ \Rightarrow \frac{\partial u}{\partial \theta} d \theta = -r F_\theta d \theta \to \]

\[ \Rightarrow \frac{1}{r} \frac{\partial u}{\partial \theta} = -F_\theta \]

\[ \Rightarrow \frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \]
Now we can show exactly in polar coordinates

\[ U(r) = \frac{1}{4\pi\varepsilon_0} \frac{q_0}{r} \]

is the correct potential for a constant force created by a
dipole \( q_0 \) at the origin on a charge \( q \)

\[ F = \mathbf{\nabla} U = -q_0 \frac{q}{4\pi\varepsilon_0 r^2} \]

\[ = -q_0 \frac{q_0}{4\pi\varepsilon_0 r^2} \]

\[ = \frac{q_0^2}{4\pi\varepsilon_0 r^2} \]  \( \Box \)

**Superposition principle for potentials**

From chapter 1 we know that the forces on a charge created
by various point charges add as

\[ \mathbf{F}_{\text{total}} = \mathbf{F}_{10} + \mathbf{F}_{20} + \ldots + \mathbf{F}_{n0} \]

\[ = \sum_{n=1}^{\infty} \mathbf{F}_{n0} \]

Suppose we know all the potentials of the various \( \mathbf{F}'s \):

\[ \mathbf{F}_{n0} = -\left( \frac{\partial U_n}{\partial x} \mathbf{i}_x + \frac{\partial U_n}{\partial y} \mathbf{i}_y + \frac{\partial U_n}{\partial z} \mathbf{i}_z \right) \]

Then since the derivative of a sum is the sum of the derivatives

We can write

\[ \mathbf{F}_{n0} = -\frac{\partial U_n}{\partial x} \mathbf{i}_x \]

\[ \Rightarrow \mathbf{F}_{n0,x} = \sum_{k=1}^{n} \mathbf{F}_{n0,k} = \frac{-n}{2\pi\varepsilon_0} \frac{\partial U_n}{\partial x} = -\frac{2}{\varepsilon_0} \sum_{k=1}^{n} \frac{U_n}{x} \]

and analogously for \( y \) and \( z \).
So 

\[ U_{tot} = \sum_{i=1}^{n} U_i \]

The potential for the total force is thus given by the sum of the potentials for the single charges.

Example: Dipole again

This time in the original coordinate system :-)

We know the potential of single charge given by

\[ U_1 = \frac{q_1 q_0}{\epsilon_0 \overline{r}_1} = \frac{q_1 q_0}{\epsilon_0 \sqrt{x^2 + (y_1 - a)^2}} \]

\[ U_2 = \frac{q_2 q_0}{\epsilon_0 \overline{r}_2} = \frac{q_2 q_0}{\epsilon_0 \sqrt{x^2 + (y_2 + a)^2}} \]

So the potential in the new coordinate is

\[ U = U_1 + U_2 = \frac{q_1 q_0}{\epsilon_0 \overline{r}_1} \left[ \frac{1}{\sqrt{x^2 + (y_1 - a)^2}} - \frac{1}{\sqrt{x^2 + (y_2 + a)^2}} \right] \]

\[ F_x = -\frac{\partial U}{\partial x} = \frac{q_1 q_0}{\epsilon_0} \left[ \frac{x}{\sqrt{x^2 + (y_1 - a)^2}} - \frac{x}{\sqrt{x^2 + (y_2 + a)^2}} \right] \]
\[ F_y = - \frac{\partial \phi}{\partial y} = \frac{q \psi_0}{4 \pi \epsilon_0} \left( \frac{y-a}{\sqrt{x^2 + (y-a)^2}} \right)^{3/2} - \frac{y+a}{\sqrt{x^2 + (y+a)^2}} \right)^{3/2} \]

For \( y > 0 \) we obtain the same result as in chpt. II.
Gauss's Law (Old. Feb. 15) (Own version)

I chose my own way of representing the material since the way in the book is quite complicated.

4.1 Statement of the Theorem

We ask the question whether we can find the charge distribution in space by measuring the electric field. This is indeed the case. To state the answer, known as Gauss's Law, we have to define a new quantity: the electric flux. It is easier to proceed if we have to define a new quantity called the flow of particles, defined by imagining a momentary jet of particles streaming through a curved surface. The density of particles per jet streaming through a given surface is defined by the flux of particles per jet streaming through the same surface. At each point in space, we also need to know the velocity of the fluid particles at this point. We assume that velocity of the fluid particles at this point is independent of time. We have a stationary flow, i.e., it is independent of time.

\[ \mathbf{A} \cdot \mathbf{n} \]

Now we ask: How many particles pass through the

\[ \text{Surface} \text{ Element} \]

\[ \mathbf{A} \text{ B} \]

\[ \mathbf{A}\mathbf{B} \text{ has the magnitude of the Surface Element and points} \]

\[ \mathbf{x} \text{ and is always} \]

\[ 1 \text{ to the Surface Element} \]
Now in a small time element of time, all the particles contained in the volume element \( dV \) stream through the surface element \( dA \). That means

\[ dN_{dA} = n(v^2) \frac{dV}{dA} = n(v^2) \frac{3}{2} \frac{dA}{dA} \frac{dA}{dV} \]

Thus in a certain time we have

\[ \frac{dN_{dA}}{dt} = n(v^2) \int dA_{dA} \int dV \]

the particles streaming through some a small element \( dA \) of the surface.

To find the total of all particles streaming through the whole surface per unit time, just

\[ \int_0^\infty \frac{dN_{dA}}{dt} = \int_0^\infty \int dA_{dA} \int dV \]

hence to sum over all surface elements:

\[ \frac{dN_{dA}}{dt} = \int_0^\infty \int dA_{dA} \int dV \]

That becomes a sum of many small \( dA \)'s in the hemisphere small

leading to a surface integral:

\[ \frac{dN_{dA}}{dt} = \int dA \int dV (v^2) \int (v^2) \]

If there is no source of fluid inside the volume, we must have

\[ \int dA \int dV (v^2) \int (v^2) = 0 \quad (no \ sources) \]

Suppose there is some source within the volume providing \( f \)

The total of all particles per unit volume at each point in space is particles per unit volume. Then

\[ \frac{dN_{dA}}{dt} \]

we have

\[ \int dA \int dV (v^2) \int (v^2) = \int dV \int \frac{dN_{dA}}{dt} = \frac{dN_{dA}}{dt} \int \frac{dN_{dA}}{dt} \]

The rate of production in \( v \)
Newly we have learnt that the sources of all fields are the electric charges of particles, we might think that the total charge contained in a volume is given by the integral over that volume. One should be specific on how one calculates, one should be careful not to think that the electric field is the force of course and think that the electric field is the force of some kind of particles. It’s just a mathematical description of some kind of particles. It’s just a mathematical description of some kind of particles.

we have always

\[ \int \nabla \cdot \mathbf{E} = \frac{Q_v}{\varepsilon_0} \]  

\( \text{Gauss' Law} \)

when \( Q_v \) is the charge contained in the volume.

V. Note that \( \varepsilon_0 \) is just a convenient factor introduced by the choice of units for the charge.

4.2 Proof for a simple point charge

(a) Point charge not contained in the volume

Put the point charge at origin of coordinate system. Then we have the following picture.
We wish to show that then
\[ \int d^2 \mathbf{E} = 0 \]

For that purpose, we draw a cone through the origin, intersecting the surface \( dS \) in two points.

The cone may be closed at the end by a suitable spherical surface element \( dS' \):

\[ dS' = dS' \cap \text{the sphere} \]

This surface element of \( dS \) is inclination to \( dS' \) by the angle

\[ \phi = \text{angle through } \text{incidence} \]

Thus the sphere is always perpendicular to the sphere. Thus

\[ dS = ND1 \cdot NAC = ND1 \cdot AB \cos \phi = \frac{1}{2} ds^2 \cos \phi \]

\[ dS' = \frac{1}{2} ds'^2 \]

Now \( dS \cos \phi = \frac{1}{2} ds^2 = \frac{1}{2} ds'^2 \)
So we have the case of
\[ d\mathbf{s}_2 \cdot \mathbf{E}^r = d\mathbf{s}_1 \cdot \mathbf{E}_r = |d\mathbf{s}_1| \mathbf{E}_r = d\mathbf{s}_1 \cdot \mathbf{E}_r \]
and that \( d\mathbf{s}_2 \) is always
pointing out of the volume.
So \( d\mathbf{s}_2 \) is oppositely directed
(towards \( q \)) than \( d\mathbf{s}_1 \).

Now:
\[ d\mathbf{s}_2 \cdot \mathbf{E}_r = -d\mathbf{s}_1 \cdot \mathbf{E}_r = \frac{-1}{4\pi \epsilon_0 \frac{q}{r^2}} \]

\[ d\mathbf{s}_2 \cdot \mathbf{E}^r = d\mathbf{s}_1 \cdot \mathbf{E}_r = \frac{1}{4\pi \epsilon_0 \frac{q}{r^2}} \]

\[ d\mathbf{s}_2 \cdot \mathbf{E}^\prime = -d\mathbf{s}_1 \cdot \mathbf{E}^\prime = -\frac{1}{4\pi \epsilon_0 \frac{q}{r^2}} \]

\[ -\frac{1}{4\pi \epsilon_0} d\mathbf{s}_1 \cdot \mathbf{E}(r_1) = -d\mathbf{s}_1 \cdot \mathbf{E}(r_1) \]

\[ = d\mathbf{s}_1 \cdot \mathbf{E}(r_1) + d\mathbf{s}_2 \cdot \mathbf{E}(r_2) = 0 \]

Now we sum over all surface elements of \( dV \) leading to

\[ \int d\mathbf{s} \cdot \mathbf{E}^2 = 0 \]
First we choose a spherical coordinate system. Then we can calculate the electric flux without much work. We shall see how to do so calculations in practice.

First we remember the vector product. Two vectors \( \mathbf{a} \) and \( \mathbf{b} \) in 3-dimensional space form a vector \( \mathbf{c} = \mathbf{a} \times \mathbf{b} \) which has the magnitude \( |\mathbf{c}| = |\mathbf{a}| |\mathbf{b}| \sin \theta \). The direction perpendicular to both the plane spanned by \( \mathbf{a} \) and \( \mathbf{b} \) and the direction of \( \mathbf{c} \) can be determined by the right-handed rule: put your thumb along \( \mathbf{a} \), and your index finger along \( \mathbf{b} \); your middle finger points in the direction of \( \mathbf{c} \):

\[
|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta
\]

The surface of a sphere can be parameterized with spherical coordinates system of a spherical coordinate system

\[
\rho = r \\
\theta = \theta \\
\phi = \phi
\]

\[
\begin{align*}
\rho &= \mathbf{r} \\
\theta &= \theta \\
\phi &= \phi
\end{align*}
\]
\[ \Phi = \oint d\Phi = R \left( \frac{\sin \theta \cos \phi}{\sin \theta} \right) \left( \frac{\sin \theta \sin \phi}{\sin \theta} \right) \left( \frac{-\cos \theta}{\cos \theta} \right) \]

\[ ds^2 = \frac{d\rho^2}{\sin^2 \theta} + d\theta^2 + \sin^2 \theta d\phi^2 \]

\[ ds = R \left( \frac{\sin \theta \cos \phi}{\sin \theta} \right) \times \left( \frac{\sin \theta \sin \phi}{\sin \theta} \right) \left( \frac{-\cos \theta}{\cos \theta} \right) \]

\[ = R^2 \left( \frac{\sin^2 \theta \cos \phi}{\sin \theta} \right) \left( \frac{\sin \theta \sin \phi}{\sin \theta} \right) \left( \frac{-\cos \theta}{\cos \theta} \right) \]

This choice of variables and order of products proves the correct notion of ds\^2 radially out of the sphere:

\[ S = \int_0^\pi \int_0^{2\pi} \sin \theta \, d\theta \, d\phi \quad ds^2 = \int_0^\pi \int_0^{2\pi} R^2 \sin \theta \, d\theta \, d\phi \]

\[ = \frac{2\pi}{\sin \theta} \left[ \sin \theta \right]_0^\pi = \frac{2\pi}{\sin \theta} \left[ -\cos \theta \right]_0^\pi = 4\pi R^2 \]

\[ \Phi = \int_0^{2\pi} d\phi \int_0^\pi \sin \theta \, d\theta \, d\phi \]

\[ = \frac{4}{4\pi} \left( \frac{1}{4\pi} R^2 \right) \]

\[ = \frac{4}{4\pi} \left( \frac{1}{4\pi} R^2 \right) \]

\[ = \frac{4}{4\pi} \left( \frac{1}{4\pi} R^2 \right) \]

\[ = \Phi \left( \frac{4}{4\pi} \right) \]
We just take out a sphere from the volume, completely closed inside it. Then the volume \( V - S_R \) does not contain the charge, and we have shown that then

\[
\int_{d(V-S_R)} dS \cdot \mathbf{E} = 0
\]

Now for the sphere we have to take the \( dS^2 \) inward, because we want to point it out of the volume \( V - S_R \). Thus writing it in the usual way with \( dS^2 \) outward, we have an additional - sign:

\[
\int_{dV} dS^2 \cdot \mathbf{E}^2 - \int_{dS_R} dS \cdot \mathbf{E}^2 = 0
\]

Or

\[
\int_{dV} dS^2 \cdot \mathbf{E}^2 = \int_{dS_R} dS^2 \cdot \mathbf{E}^2 = \frac{Q}{\varepsilon_0}
\]

This proves Gauss's law for a point charge.
4.3 Proof Warding any charge distribution

(a) Spherical point charge

That's easy since

\[ E^2_{\text{total}} = \sum_{q=1}^{n} E^2_q \]

With \( E^2_q = \frac{\kappa_2}{4\pi\varepsilon_0\left(\kappa_2 - \kappa_1\right)^3} \)

The field of charge \( q_r \).

Thus:

\[ \Phi_{\text{total}} = \int dS^2 \sum_{q=1}^{n} E^2_q \]

\[ = \sum_{q=1}^{n} \int dS \cdot E^2_q = \sum_{k=1}^{n} \Phi_{q_k} \]

No. 1 \( \Phi_{q_k} = \frac{\kappa_2}{4\pi\varepsilon_0} \) if \( q_k \) is inside \( V \)

No. 1 \( \Phi_{q_k} = 0 \) if \( q_k \) is outside \( V \)

\[ \Rightarrow \Phi_{\text{total}} = \int dS^2 E^2_{\text{total}} = \sum_{\text{all } k} \frac{q_k}{4\pi\varepsilon_0} \]

(b) Arbitrary volume charge

\[ E^2 = \int dV \left( \frac{G \left( q^2 \right)}{4\pi\varepsilon_0 \left( \kappa_2 - \kappa_1 \right)^3} \right) \]

\[ \Rightarrow \Phi_{\text{total}} = \int dS^2 \int dV \left( \frac{G \left( q^2 \right)}{4\pi\varepsilon_0 \left( \kappa_2 - \kappa_1 \right)^3} \right) \]

\[ \Rightarrow \Phi_{\text{total}} = \sum_{\text{all space}} \frac{q_k}{4\pi\varepsilon_0} \]
\[ \Phi^{(2)} = \int \text{d} V ^{1} g (r^{2}) \oint \frac{\mathbf{E} \cdot \mathbf{n}}{\varepsilon_{0}} \frac{r - r^{1}}{r^{2} - r^{1} \sqrt{3}} \]

Let the surface integral be the electric flux of a point charge with \(q = 1\), i.e., this gives
\[ \Phi^{(2)} = \int \text{d} V ^{1} g (r^{2}) \chi (r^{1}) = \int \text{d} V ^{1} g (r^{2}) = \frac{q V}{\varepsilon_{0}} \]

where
\[ \chi (r^{2}) = \begin{cases} 1 & \text{if } r^{2} \in V \\ 0 & \text{if } r^{2} \notin V \end{cases} \]

and \(q V\) is the charge on side of \(V\). \(\square\)
\[ F = m \frac{v^2}{r} = q \mathbf{E}(r) \]

Motion of a particle with mass \( m \) in an electrostatic field. We know that \( \mathbf{E} \) is conservative and thus also the force:

\[ F(r) = q \mathbf{E}(r) \]

is conservative, i.e., there exists a potential \( U \):

\[ F = - \left( \frac{\partial U}{\partial x} \mathbf{i}_x + \frac{\partial U}{\partial y} \mathbf{i}_y + \frac{\partial U}{\partial z} \mathbf{i}_z \right) \]

(Sum in Cartesian coordinates).

Since the electric potential \( U \) is defined by:

\[ \mathbf{E} = - \left( \frac{\partial U}{\partial x} \mathbf{i}_x + \frac{\partial U}{\partial y} \mathbf{i}_y + \frac{\partial U}{\partial z} \mathbf{i}_z \right), \]

we have the relation:

\[ U = q \mathbf{E} \]

**Energy Conservation Law**

With a conservation law we have:

\[ m \mathbf{\ddot{r}} = - \left( \frac{\partial U}{\partial x} \mathbf{i}_x + \frac{\partial U}{\partial y} \mathbf{i}_y + \frac{\partial U}{\partial z} \mathbf{i}_z \right) \]

Suppose we have solved the equation of motion. Then we have:

\[ m \mathbf{\ddot{r}} = \frac{d}{dt} \left( \frac{m}{2} \mathbf{\dot{r}}^2 \right) = - \left( \frac{dU}{dx} \mathbf{i}_x + \frac{dU}{dy} \mathbf{i}_y + \frac{dU}{dz} \mathbf{i}_z \right) \]

\[ = - \frac{d}{dt} \left( U[r(t)] \right) \]

or:

\[ \frac{d}{dt} \left( \frac{m}{2} \mathbf{\dot{r}}^2 + U[r(t)] \right) = 0. \]
This means that

\[ E = \frac{1}{2} m \dot{q}^2 + U(q(t)) \]

is constant during the whole motion of the particle. We can use the formula \( E = \frac{1}{2} m \dot{q}^2 + U(q) \) for two points on the path where \( t = t_1 \) or \( t = t_2 \), we have

\[ m \dot{q}^2 + U(q_1) = \frac{1}{2} m \dot{q}_2^2 + U(q_2) \]

when \( q_1 = q(t_1) \) and

We also know that due to the existence of an potential, its value depends only on the initial and final point and not on the specific path which connects them:

\[ U(q_2) - U(q_1) = \int_{C_{q_1-q_2}} d \dot{q} \, \dot{q} \]

Especially with the work done on the particle by the force \( F \) we have

\[ W = \int_{C_{q_1-q_2}} d \dot{q} \, \dot{q} = \int_{C_{q_1-q_2}} d \dot{q} \, \dot{q} = [q(t) - q(t_0)] \]

or for electric field

\[ W = -q \left[ V(q(t)) - V(q(t_0)) \right] \]

Examples from book

Chapter II Problem 6 (Wire, energy law)

\[ E = -EX \]

\[ V = EX \]
Initial state: \( x = 0 \), \( v_x = v_0 \)

\[ E = \frac{1}{2} m v_0^2 \]

Energy, because \( E \) is residual (without field).

Final state: \( x = L \), \( v_x = 0 \)

\[ E = q E L \]

\( \Rightarrow \frac{m}{2} v_0^2 = q E L \) \( \Rightarrow \) \( E = \frac{m v_0^2}{2 q L} \)

Exercise 5: Exercise \( 1 \) \( \Rightarrow \) solutions to exercises

Exercise 6: With energy law

\[ E = -E \delta_x \) \( \Rightarrow V = E \]

\[ E = 0 = \frac{m}{2} v^2 - e E \]

\[ E = 0 = \frac{1}{2} m v^2 - e E \]

\( \Rightarrow v = \sqrt{\frac{2 e E}{m}} = 3.25 \times 10^5 \frac{m}{s} \)

III. Problems 2, 4; Exercise 16 (5, 6) \( \Rightarrow \) see solutions to exercises