13. Maxwell's equations and electromagnetic waves

13.1 The displacement current

We look again at the RC circuit.

\[ V + iC \frac{dq}{dt} + iRq = 0 \]

We used Faraday's law to find the current as

\[ \oint E \cdot ds = \nabla \times B = 0 \]

where we have neglected the self-inductance of the loop. Further we have

\[ i = \frac{dq}{dt} \]

and thus

\[ R \frac{dq}{dt} + \frac{q}{C} = V \]

The solution is

\[ q = CV + A \exp \left( -\frac{t}{RC} \right) \]

with \( A = C V + 2 \exp \left( -\frac{t_0}{RC} \right) \)

and

\[ i = \frac{dq}{dt} = \frac{V}{R} \exp \left( -\frac{t}{RC} \right) \]
According to Ampere's circuital law, the current leads to a B field, where

$$\oint ds \cdot \mathbf{B} = \mathbf{M}_0 \cdot i$$

where $ds$ is the boundary of an arbitrary surface through which the current $i$ flows. Now we look at the upper plate of the capacitor.

Let $S$ be a disk of radius $r$ with the circle on which $ds$ is going in direction of the current. Then

$$\oint ds \cdot \mathbf{B} = B \ 2\pi r = M_0 \ i$$

Now we look at the following surface.

This dome-like surface also has $ds$ as a boundary, but on the other hand there is no current passing through it. This leads to the contradiction

$$\oint ds \cdot \mathbf{B} = 0 \ \text{but} \ \oint ds \cdot \mathbf{B} = M_0 \ i$$

$$\oint ds$$
Maxwell's idea to solve the problem was Ampère's law over the observation that there is a time-changing electric field $\mathbf{E}$ between the plates of the capacitor and is give by Gauss's law

$$\mathbf{E}^2 = \frac{Q}{\varepsilon_0} \left( -\frac{dA}{dt} \right)$$

Thus the flux of $\mathbf{E}^2$ through $S$ is

$$\Phi_E = \int_S dA \cdot \mathbf{E}^2 = \frac{Q}{\varepsilon_0}$$

when only the plates below are conducting a current $\mathbf{i}$.

Now on the other hand, we have

$$\mathbf{i} = \frac{dQ}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt}$$

Thus, we can save Ampère's law by adding $\frac{d\Phi_E}{dt}$ to the current. This is called the displacement current.

We can also write $\mathbf{B}$ in the local form

$$\oint_S dA \cdot \mathbf{B} = \mu_0 \oint_S dS \left( \mathbf{E}^2 + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

The orientation of $\mathbf{B}$ is the current density $\mathbf{j}$, and $\mathbf{S}$ and $dS$ have to be related by the right-hand rule.

Also, the current conservation law is fulfilled, if we
Consider the displacement current in the law. Then show the complete hemispherical $S$, enclosing the capacitor plate
\[ \frac{\partial}{\partial t} \int S \left( \varepsilon_0 \varepsilon \right) d\mathbf{E}^2 = -i + \varepsilon_0 \frac{d}{dt} \varepsilon = 0, \]

because according to Maxwell's law
\[ \int S \left( \varepsilon_0 \varepsilon \frac{d\mathbf{E}}{dt} \right) d\mathbf{V} = \int V \varepsilon_0 \varepsilon \frac{d\mathbf{E}}{dt} d\mathbf{V} = \varepsilon_0 \frac{d}{dt} \int V \varepsilon_0 \varepsilon d\mathbf{V} \]

or
\[ \frac{d\mathbf{V}}{dt} = 0. \]

In other words
\[ \int S \left( -\frac{d\mathbf{V}}{dt} \right) = -\frac{d\mathbf{V}}{dt} V = -\frac{d}{dt} \int V d\mathbf{V} \]

when $d\mathbf{V} = V d\mathbf{S}$ and the orientation of $d\mathbf{V}$ is always out of the volume.

13.2 The complete set of Maxwell's Equations

We can now write all our knowledge about electromagnetism in terms of linear equations

1. $\int V \varepsilon_0 \varepsilon = \frac{1}{\varepsilon_0} \int d\mathbf{V} / \mathbf{V}$ (Faraday's law) \( \Rightarrow \) electromotive force is the source of an electric field

2. $\int V \varepsilon_0 \varepsilon = 0$ (No magnetic lines of force)
\[ \mathbf{E} \cdot \mathbf{j} = - \frac{d}{dt} \int_S \mathbf{E} \cdot \mathbf{n} \, dS \]  

(Technique's law)

\[ \int_S d\mathbf{S} \mathbf{B} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{B} \, dS \quad \text{(Ampère's law)} + \text{Maxwell's displacement current}
\]

Here always

- Boundaries \( \partial V \) of a volume \( V \) are closed surfaces with the normal vectors pointing outwards of the volume.

- The direction of the boundaries \( dS \) of a surface \( S \) (which are closed lines) determines the direction of the normal vector of this surface \( S \), according to the right-hand rule:

  Giving the thumb of the right hand the direction of the boundary \( dS \), the fingers point in direction of \( d\mathbf{S} \).

13.3. Current conservation again.

If we use a closed surface \( S \) in (14), the left-handed such as \( \mathbf{0} \), because a closed surface has no boundary:

\[ \int_S d\mathbf{S} \mathbf{B} \left( \mathbf{j} + \varepsilon_0 \frac{d\mathbf{E}}{dt} \right) = 0 \]

\[ \oint_{\partial V} \mathbf{B} = \mathbf{0} \]

\( \partial V \) is the boundary of the volume \( V \) closed by the surface \( S \).

\( \mathbf{B} \) is always
If the surface is in the fixed a frame (not moving in any way), then we can write
\[ \oint \frac{d\mathbf{E}}{dt} = \frac{1}{c^2} \oint \frac{d\mathbf{E}}{dt} \, dV \]

Thus:
\[ \oint \frac{d\mathbf{E}}{dt} \, dV = -\frac{d}{dt} \oint \mathbf{E} \, dV \]

Any charge flowing out of the boundary of a volume must come from the a sort of this volume. The total amount of charge must be conserved.

13.4. Electromagnetic waves

The most important result from Maxwell's completion of the laws of electromagnetism is his prediction of electromagnetic waves, i.e., electromagnetic fields which propagate through space without waves and charges.

For this case, Maxwell's equations read
\[ \oint d\mathbf{E} = 0 \quad \text{(no charges)} \]
\[ \oint d\mathbf{B} = 0 \quad \text{(no magnetic monopoles)} \]
\[ \frac{\delta}{\delta s} E^2 = - \frac{d}{dt} \int ds \cdot \mathbf{B} \]

\[ \frac{\delta}{\delta s} B^2 = \mathbf{E}_0 \cdot \frac{d}{dt} \int ds^2 E^2 \]

Now we wish to find the most simple set of fields \( E \) and \( B \) to obey this set of equations. We assume that

\[ E^2 = E_y(t,x)^2 \]

We start with Faraday's law for a small rectangle in the \( xy \) plane:

\[ \oint \mathbf{E} \cdot d\mathbf{s} = \int [E_y(t, x_0 + dx) - E_y(t, x_0)] \, dy \]

\[ \frac{\delta}{\delta s} E^2 = \int [E_y(t, x_0 + dx) - E_y(t, x_0)] \, dy \]

\[ = - \frac{1}{\mathbf{E}_0^2} B_z(t, x_0) \, dx \, dy \]

\[ \Rightarrow \quad \frac{E_y(t, x_0 + dx) - E_y(t, x_0)}{dx} = - \frac{\partial B_z(t, x_0)}{\partial t} \]

For \( dx \rightarrow 0 \) we can find

\[ \frac{\partial E_y}{\partial x} = - \frac{\partial B_z}{\partial t} \quad (M) \]
We assume that
\[ B^2 = B_Z(x) \]

In Ampère-Maxwell's law, we use the path
\[ \gamma \]
\[ \mu_0 \varepsilon_0 \frac{\partial B}{\partial t} = \frac{1}{c^2} \frac{\partial^2 E_y}{\partial x^2} \]

\[ \oint \mathbf{B} \cdot d\mathbf{l} = \int \mathbf{J} \cdot d\mathbf{s} \]

\[ \int d\tau B^2 = \left[ B_Z(x_0) - B_Z(x_0 + dx) \right] dx \]

\[ = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} \]

\[ \Rightarrow \quad - \frac{\partial B_Z}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} \]

\[ \Rightarrow \quad \frac{\partial^2 B_Z}{\partial t^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} \]

\[ \Rightarrow \quad \frac{\partial^2 E_y}{\partial x^2} = - \frac{\partial^2 B_Z}{\partial x \partial t} = - \frac{\partial^2 B_Z}{\partial t \partial x} \]

Plugging this into (c) we find

\[ \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} = \frac{\partial^2 E_y}{\partial x^2} \quad \text{with} \quad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \]
This is the wave equation. It is easy to see that for any waveform \( f \),

\[
E_g(t, x) = f(\alpha x - \omega t)
\]

is a solution, as only need to determine a relation \( \alpha \) the constants \( \omega \) and \( \alpha \):

\[
\frac{\partial E_g}{\partial t} = -\omega \frac{\partial f}{\partial (\alpha x - \omega t)} \\
\frac{\partial^2 E_g}{\partial t^2} = -\omega^2 \frac{\partial f}{\partial (\alpha x - \omega t)} \\
\frac{\partial E_g}{\partial x} = \alpha \frac{\partial f}{\partial (\alpha x - \omega t)} \\
\frac{\partial^2 E_g}{\partial x^2} = \alpha^2 \frac{\partial f}{\partial (\alpha x - \omega t)}
\]

Plugging this into the wave equation gives you

\[
\left( \frac{\alpha^2}{C^2} - \omega^2 \right) \frac{\partial^2 f}{\partial (\alpha x - \omega t)} = 0
\]

which is solved by relating \( \mu \) and \( k \) to make the bracket vanish

\[
\frac{\omega^2}{C^2} \quad \text{or} \quad k = \pm \frac{\omega}{C}
\]

Thus we hand here a more general solution, namely

\[
E_g(t, x) = f(\alpha x - \omega t) + f(\alpha x + \omega t)
\]

which describes the superposition of waves running in the positive \(( k = + \frac{\omega}{C} )\) and negative \(( k = - \frac{\omega}{C} )\) \( x \) direction.

Particularly important are waves with a fixed oscillation frequency, for a wave running in \( +x \) direction this is:

\[
E_g(t, x) = A \cos(\alpha x - \omega t) \quad \text{with} \quad k = + \frac{\omega}{C}
\]
\[ E_y(x,t) = A \cos \left( \frac{x}{\lambda} - \omega t \right) \]

The speed of waves is given by

\[ \omega T = 2\pi \Rightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{\lambda} \] (wave frequency)

and the space by

\[ \frac{2\pi}{\lambda} = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{\lambda} \] (wave length)

From \( \lambda = \frac{c}{f} \) we have

\[ \frac{2\pi}{\lambda} = \frac{2\pi}{cT} \Rightarrow \lambda = cT \]

The speed of the wave thus moves with the velocity \( c \).

This has been determined to be

\[ c = \sqrt{\frac{1}{\mu_0 \varepsilon_0}} \]

Now \( \varepsilon_0 = \frac{1}{4\pi \cdot 9 \times 10^9} \frac{C^2}{Nm^2} \nu \sec^2 \) \Rightarrow \( c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \cdot 10^8 \text{ m/s} \)

\( \mu_0 = 4\pi \cdot 10^{-7} \frac{N\text{m}^2}{C^2} \)

\( c \) is the speed of light and Max Planck to the conclusion that light is an electromagnetic wave.

The magnetic field is now determined by Eq. (47)

\[ \frac{\partial B_x}{\partial t} = - \frac{\partial E_y}{\partial x} = A \lambda \sin (kx - \omega t) \]

Up to a constant B-field this is...
\[ B_z = \frac{\Phi x}{w} \cos (hx - wt) \]

\[ B_z = \frac{4}{c} \cos (hx - wt) \]

The \( B \)-field is thus perpendicular to \( E \), both \( E \) and \( B \)

The remaining Maxwell equations hold in the case that there

For \( V \) we choose a small box of \( x_0^2 \):

\[ \int dS^2 \cdot E^2 = \left[ E_y(t, y_0) + d y \left( \frac{E_y}{y} \right) - E_y(t, y_0) \right] dx dy = 0 \]

In case \( E_y \) does not depend on \( y \), only on \( x \).

For the \( B \)-field we have

\[ \int dS^2 \cdot B^2 = [B_z(6, y_0, z_0) - B_z(t, y_0)] dx dy = 0 \]

Since \( B \) does not depend on \( z \).
Characteristics of “free” em. Waves

- Electric and magnetic fields oscillate \perp to direction of propagation \Rightarrow \textbf{transverse waves}
- \( \vec{E} \perp \vec{B} \) are in phase
- Phase velocity: \( c = 1/\sqrt{\mu_0 \epsilon_0} = \text{speed of light} \)
- Dispersion relation: \( \omega = 2\pi f = c|\vec{k}| \) or \( \lambda = cT \) (\( |\vec{k}| = 2\pi/\lambda \)).
- Sources (not explained in this course)
  -Accelerated charged particles =
  -\textbf{time-dependent} electric charges and currents
  -Modern picture: quantum-mechanical transitions in atoms (visible light, UV) and nuclei (\( \gamma \) rays)

The em. spectrum

\[\begin{array}{c|c|c|c|c|c|c|c|c|c}
\hline
\text{Frequency (Hz)} & \text{Visible spectrum} & \text{IR} & \text{Microwave} & \text{FM Radio waves} & \text{AM Radio waves} & \text{Long radio waves}
\hline
\text{Increasing Frequency (\( \nu \))} & \text{Increasing Wavelength (\( \lambda \))} & \text{Visible spectrum} & \text{IR} & \text{Microwave} & \text{FM Radio waves} & \text{AM Radio waves} & \text{Long radio waves}
\hline
\end{array}\]
### The em. spectrum

<table>
<thead>
<tr>
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<th>$f$ (Hz)</th>
<th>$\lambda$ (m)</th>
<th>source (ex.)</th>
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<tbody>
<tr>
<td>$\gamma$ rays</td>
<td>$&gt; 10^{20}$</td>
<td>$&lt; 10^{-12}$</td>
<td>radioactivity, nuclear transitions</td>
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<tr>
<td>X rays</td>
<td>$&gt; 3 \cdot 10^{16}$</td>
<td>$&lt; 10^{-8}$</td>
<td>bremsstrahlung radiation, atomic transitions</td>
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<tr>
<td>UV</td>
<td>$7.5 \cdot 10^{14} - 3 \cdot 10^{16}$</td>
<td>$4 \cdot 10^{-7} - 10^{-8}$</td>
<td>our Sun</td>
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<tr>
<td>visible light</td>
<td>$4 \cdot 10^{14} - 7.5 \cdot 10^{14}$</td>
<td>$4 \cdot 10^{-7} - 7.5 \cdot 10^{-7}$</td>
<td>atomic transitions</td>
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<tr>
<td>infrared</td>
<td>$3 \cdot 10^{11} - 4 \cdot 10^{14}$</td>
<td>$10^{-3} - 7.5 \cdot 10^{-7}$</td>
<td>transitions between vibrational modes of molecules</td>
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<tr>
<td>Millimeter Waves</td>
<td>$30-300 \cdot 10^9$</td>
<td>$10^{-3}-10^{-2}$</td>
<td>antenna</td>
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<tr>
<td>microwaves</td>
<td>$1.6-30 \cdot 10^9$</td>
<td>$10-187 \cdot 10^{-3}$</td>
<td>magnetrons (microwave oven), rotation and torsion transitions of molecules, cosmic microwave background</td>
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<tr>
<td>radio waves</td>
<td>$5 \cdot 10^5-1.6 \cdot 10^9$</td>
<td>$0.2-200$</td>
<td>antenna</td>
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